ASPECTS OF SUPERSYMMETRIC MECHANICS

by

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Submitted to the Institute for Graduate Studies in Science and Engineering in partial fulfillment of the requirements for the degree of Master of Science

Graduate Program in Physics

Boğaziçi University

2015

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DATE OF APPROVAL: 23.12.2014

ACKNOWLEDGEMENTS

First and foremost I owe my sincerest gratitude to my supervisor, Assoc. Prof. Dieter Van Den Bleeken , who has guided me throughout writing of my thesis with his patience, motivation and knowledge. He had an organized schedule for meeting every week and always had time to answer my questions about thesis. I am deeply grateful to him for the excitement regard to teaching, useful comments and remarks. I would like to thank him for funding me along the thesis by TÜBİTAK (project no. 113F164). I am also thankful to him for encouraging the use of correct grammar and consistent notation in my writings and for carefully reading and commenting on countless revisions of this text. I attribute the level of my Masters degree to his encouragement, effort and guideness.

Besides my advisor, I would like to thank the rest of my thesis committee: Prof. Osman Teoman Turgut, Prof. Ali Kaya and Prof. Nihat Sadık Değer for their insightful comments and questions.

ABSTRACT

ASPECTS OF SUPERSYMMETRIC MECHANICS

Supersymmetry is a space time symmetry which relates bosons to fermions or vice verca. It requires that for each particle there has to be an anti- particle with the same mass. Along this thesis some supersymmetric Lagrangian models and their properties are discussed. One of them is a Quiver Lagrangian model of the classical system of D-particles connected to each other by light strings. Quiver mechanics is used in a quantum description of black holes. The Quiver quantum mechanical model is one of the key ways to understand the thermodynamic properties of large N=2 black holes in 4 dimensions from the point of view of string theory. In the thesis some quantum approximations are given to have an effective Quiver Lagrangian similar to the Lagrangian in a background magnetic field with a Dirac monopole term. Following it, we discuss the classical properties of the Higgs and Coulomb branches of N=4 Quiver mechanics and derive Coulomb and Higgs minima of vacua in this thesis. However, we are addressing the question whether a stable Coulomb Branch can also be obtained, classically. We use separation of scales and quasi-classical expansion methods to derive the same effective Lagrangian in a classical way. Separation of scales is the method of determining the fast and slow fields in the Lagrangian and eliminating the effect of fastly oscillating fields by averaging them in a long time interval. We aim to get some time independent effective potentials for slowly changing fields by using the adiabatic invariant theorem. Quasi-classical expansion describes the bosonic classical dynamics along with fermionic degrees of freedom. It tells us that the classical solution always involves Grassmann terms with a quasi-classical solution when the coupling between bosons and fermions appears in the equations of motion. Therefore, it is a Grassmann valued function.

ÖZET

SÜPERSİMETRİK MEKANİĞİN YÖNLERİ

Süpersimetri boson ile fermiyon arasında dönüşüm ilişkisi kuran bir uzay zaman simetrisidir. Süpersimetriye göre her parçacık için aynı kütleye sahip bir zıt parçacık olmalıdır. Tez boyunca bazı süpersimetrik Lagrange modelleri ele alınmış ve özellikleri incelenmiştir. Bunlardan biri de sicimlerle birbirlerine bağlanmış D-parçacıklarn oluşturduğu klasik sistemin Quiver Lagrange modelidir. Quiver(okluk) mekaniği karadeliklerin quantum mekaniksel yapısını açıklamada kullanılır. Bu model 4 boyutta N=2 kara deliklerin termodinamik özelliklerini anlamak için kullanılan önemli modellerden birisidir. Quiver sistemini tanımlayan Lagrange denklemi kuantum mekanik yaklaşımlarla, manyetik alan içerisindeki bir parçacığın mekaniğini gösteren ve Dirac monopol terimi içeren Lagrange'a benzer bir efektif Lagrange denklemine dönüşür. Bu tezde N=4 Quiver mekaniğinin Higgs ve Coulomb Branch'lerinin klasik özellikleri tartışılmış ve vakum durumu için Higgs ve Coulomb minimum noktaları bulunmuştur. Bunun yanı sıra, stable Coulomb branch'in klasik yaklaşımlarla elde edilip edilemeyeceği sorusuna yanıt aranmıştır. Quantum mekaniksel yaklaşımlarla elde edilen efektif Lagrange denkleminin klasik mekaniksel yaklaşımlarla da bulunabilmesinin mümkün olup olmadığını öğrenmek için Separation of scales ve quasi-klasik açılım metodları uygulanmıştır: Separation of scales, Lagrange denkleminde yavaş ve hızlı değişen alanları belirleyip hızla salınan alanın sistem üzerindeki etkilerini uzun bir zaman aralığında ihmal etmeyi öngören bir metottur. Yavaşça değişen alanlar için ise adiabatik invariant teoremini kullanarak zamandan bağımsız efektif potansiyeller elde etmek amaçlanmıştır. Quasi-klasik açılımda bir alanın bozonik klasik dinamiği fermiyonik serbestlik dereceleri ile birlikte tanımlanır. Klasik çözüm, hareket denklemlerinde bozon ve fermiyon çifti olduğunda, quasi-klasik ve Grassmann terimleri cinsinden yazılabilir. Bu nedenle bir Grassmann fonksiyonudur.

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LIST OF SYMBOLS

A	Gauge Potential
$\mathbf{A}$	Magnetic Vector Potential
$\mathbf{B}$	Magnetic Field
D	Bosonic Auxilliary Field
$\mathbf{E}$	Electric Field
F	Fermionic Auxilliary Field
g	Magnetic Charge
G	Grassmann Algebra
H	Hamiltonian
I	Adiabatic Invariant
L	Lagrangian
m	Mass
M_{ij}	Component of The Matrix
$M_{(ij)}$	Symmetric Matrix
$M_{[ij]}$	Antisymmetric Matrix
n	Mixture of Boson and Fermion
p	Electric charge
$\mathbf{p}$	Bosonic Momentum
q	Boson
Q	Super charge
S	Action
w	Frequency
α	Matrix
δ	Variation
ϵ	Complex Grassmann variable
θ	Grassmann Number
κ	Charge quantization
λ	Spinor

μ	Reduced Mass
ξ	Spinor
π	Fermionic Momentum
$\boldsymbol{\sigma}$	Pauli Spin Matrice
ϕ	Complex Scalar Field
Φ	Modified Complex Scalar Field
χ	Modified Matrix
ψ	Fermion
Ψ	Modified Fermionic Field
$\mathbf{A}^d$	Dirac Monopole
$\mathbb{Z}_2$	$\mathbb{Z}_2$ Graded Lie Algebra
ℓ	Angular Momentum Density
$\mathbb{P}$	Momentum Density

LIST OF ACRONYMS/ABBREVIATIONS

Det	Determinant of A Matrix
qc	Quasi-Classical
QM	Quantum Mechanics
SUSY	Supersymmetry

1. INTRODUCTION

Particles are divided into bosons (of integer spin) and fermions (of half-integer spin). Supersymmetry is a symmetry relating these two kinds of particles. A supersymmetry transformation turns a bosonic state into a fermionic state, and vice versa. The super generators Q obey an anti-commutation:

$$Q|\text{Boson} >= |\text{Fermion} >, \quad Q|\text{Fermion} >= |\text{Boson} > .$$

Because they are fermionic operators, they carry spin angular momentum 1/2 and change the spin of a particle with its space-time properties. Therefore, supersymmetry is not an internal symmetry but a space-time symmetry.[1]. Up to now, there is no experimental evidence of supersymmetry in nature. Research for weak scale supersymmetry has been going on at colliders at LHC experiments, ATLAS and CMS. If it existed, it would be a spontaneous symmetry since we don't observe anti particle of each particle experimentally. But, it has a strong theoretical support which is believed to solve many outstanding problems in physics. Supersymmetry extends the Poincare group to a larger space- time symmetry (the Haag - Sohnius - Lopuszanski theorem) with the contribution of anti-commutation property of fermions. It is a leading candidate to explain the physics beyond the standard model. It can provide a solution to the hierarchy problem that is the mass of the Higgs particle is smaller than the mass of the Planck mass $m_{\text{weak}} \ll m_{\text{planck}}$ or in other words, the experimental value of the Higgs mass is smaller than its theoretical value. Supersymmetry may also explain the existence of mysterious dark matter particles [2]. It allows the unification of the Standard Model forces ; provides a connection between the SM forces and gravity; and is a candidate for the WIMP dark matter when the neutralino is the lightest supersymmetric particle (LSP). It provides a solution to the baryon asymmetry of the universe and is a candidate for the cosmological constant. For quantum gravity which is the unification of quantum mechanics and general relativity, supersymmetry is used as a mathematical tool to simplify equations in string theory and allows certain terms to cancel. Without supersymmetry the equations give some non-physical results such as imaginary energy and infinite values. Therefore, it carries great hope for future problems.

Quiver mechanics is used in a quantum description of black holes with string theory and some quantum approximations are given to have an effective Lagrangian in [19] similar to the Lagrangian with a Dirac monopole. Following it, we deduce the Higgs and Coulomb minima of zero energy ground state in this thesis. However, we claim that a stable Coulomb Branch can also be obtained, classically.

1.1. OUTLINE

The thesis is organized as follows:

In Chapter 2, we try to give a fairly complete presentation of the $\mathbb{Z}_2$ Graded Lie Algebras associated with the Grassmann algebra. Bosons take zero grades while fermions are graded as one. Commutation or anti- commutation property is determined by grading. We give some grading examples which will be useful along the thesis.

In Chapter 3, we outline the basic properties of the Grassmann algebra. Grassmann algebra is a $\mathbb{Z}_2$ Graded Lie Algebra with fermionic and bosonic subspaces. Bosons are even Grassmann numbers while fermions are odd. Fermions obey anti -commutation rule so this gives rise to finite terms in Grassmann expansion.

In Chapter 4, we review classical mechanics with the addition of fermions. We study the generalized classical Poisson bracket, symmetries and Noether theorem. We give some Lagrangian examples for fermions and mixed cases. quasi-classical expansion is introduced, which means a classical bosonic or fermionic field can be expressed by a quasi-classical solution and fermionic terms. Along the thesis, we will be using quasi-classical expansion most of the time for the following chapters since the Lagrangian with the interaction terms always has a fermionic contribution. We review the adiabatic invariant for a harmonic oscillator and apply the adiabatic invariant formula for a fermionic system, independently.

In Chapter 5, we study the simple Lagrangian model with mixed degrees of freedom and check for the supersymmetry under some given transformations. Supercharges are the conserved quantities related to supersymmetry and they form an algebra by Poisson brackets and infinitesimal fermionic transformations.

In Chapter 6, we introduce a Lagrangian for a D-particle in a background magnetic field. In particular, the conditions for the supersymmetric Lagrangian are studied independently in detail . We try to find solutions which preserve the supersymmetry and give a description of spontaneous symmetry breaking.

In Chapter 7, we study the supersymmetric Quiver mechanics. A Quiver is a diagram with nodes and arrows. Each node corresponds to a vector multiplet and each arrow corresponds to a chiral multiplet. There is a unique supersymmtric Lagrangian for them. We search for the minima of the potential that gives us Higgs and Coulomb branch conditions. Solutions that preserve the supersymmetry are studied, the supersymmetric ground state is provided by the Higgs branch.

In Chapter 8, we mention about separation of scales. Quantum mechanically chiral multiplet is eliminated from the Lagrangian and an effective Lagrangian is obtained. We are searching for the possibility of having the same effective Lagrangian by classical approximations. Our assumption is that time scale fluctuations of the chiral multiplet is much much smaller than the time scale of the vector multiplet. We consider the vector multiplet in the Lagrangian as a constant and the other chiral modes oscillate fastly. We try to find a time independent effective Lagrangian which gives the same equation of motion for slowly changing vector multiplet in time. We use both the adiabatic invariant theorem and separation of scales for two separate Lagrangian systems with a scalar coupled to a scalar and a fermionic field coupled to a scalar. We obtain constant effective potentials and they are similar to the potentials obtained by quantum approximations. For the most general Lagrangian with the interaction terms we use a quasi-classical expansion. We obtain linearly independent equations that can be solved but we come across with the secular terms that the analysis are more complicated. So we leave the remaining calculations for future work.

Finally, in the Appendix we broadly outline some mathematical tools and derivations along the thesis. We review some related subjects such as the Dirac magnetic monopole and charge quantization.

2. $\mathbb{Z}_2$ GRADED LIE ALGEBRAS

In supersymmetry since we study with bosons and fermions, we must know the algebra they generate. It is a $\mathbb{Z}_2$ graded algebra with bosons and fermions as generators. Graded Lie algebras differ from Lie algebras since they use anti-commutation relations instead of commutation relations. We say "graded" algebra because particles are graded mod 2. We follow [3] in this chapter. Now, we will give some general properties of this algebra related to this thesis:

A $\mathbb{Z}_2$ graded Lie algebra consists of a vector space that is the direct sum of two subspaces H_0 and H_1 :

$$H = H_0 \oplus H_1 \tag{2.1}$$

and a product $* : H \times H \to H$

$$f_i * g_j := f_i g_j - (-1)^{|g_j||f_i|} g_j f_i \tag{2.2}$$

with the following properties for all $f_i \in H_i$, $g_j \in H_j$, $h_k \in H_k$ and $i, j, k = 0, 1$:

(i) $f_i * g_j \in H_{i+j \mod 2}$
(ii) $f_i * g_j = -(-1)^{|f_i||g_j|} g_j * f_i$
(iii) $f_i * (g_j * h_k)(-1)^{|f_i||h_k|} + g_j * (h_k * f_i)(-1)^{|g_j||f_i|} + h_k * (f_i * g_j)(-1)^{|h_k||g_j|} = 0$

where $|..|$ denotes the grades of the elements that can only take 0 and 1 values in $\mathbb{Z}_2$ graded algebra. The elements with grade 0 and 1 are called boson and fermion and they generate the subspaces H_0 and H_1, respectively. When the order of two elements is changed, sign differs depending on the bosonic and fermionic characters of the elements:

$$f_i g_j = (-1)^{|f_i||g_j|} g_j f_i. \tag{2.3}$$

If at least one of f_i and g_j is boson, sign doesn't change because the multiplication of the grades is zero, which is the ordinary commutation between bosons or boson and fermion. However; if they are both fermions, it takes a minus sign. This is compatible with the anti-commutation relation of fermions. Now, consider the product separately on the two subspaces H_0 and H_1:

(i) $* : H_0 \times H_0 \longrightarrow H_0$, and $f_0, g_0 \in H_0$. Then the product of two bosons is

$$f_0 * g_0 = f_0 g_0 - (-1)^{|f_0||g_0|} g_0 f_0 = f_0 g_0 - g_0 f_0 = [f_0, g_0].$$

It is antisymmetric and obeys commutation relation.

(ii) $* : H_0 \times H_1 \longrightarrow H_1$ and $f_0 \in H_0$ and $g_1 \in H_1$ then we get

$$f_0 * g_1 = f_0 g_1 - (-1)^{|f_0||g_1|} g_1 f_0 = f_0 g_1 - g_1 f_0 = [f_0, g_1].$$

The product of a boson and fermion is still antisymmetric and obeys commutation relation.

(iii) $* : H_1 \times H_1 \longrightarrow H_0$ let $f_1, g_1 \in H_1$ and we have the product of two fermions:

$$f_1 * g_1 = f_1 g_1 - (-1)^{|f_1||g_1|} g_1 f_1 = f_1 g_1 + g_1 f_1 = \{f_1, g_1\}$$

Contrary to the previous products, it is symmetric and obeys an anti-commutation relation.

2.1. Examples for Grading

The grading of a multiplication in $\mathbb{Z}_2$ is the sum of each grades:

$$|f_i g_j| = |f_i| + |g_j| \quad i, j = 0, 1. \tag{2.4}$$

When we multiply two bosons or two fermions, they behave like a boson with zero grade. However, the multiplication of boson and fermion is fermionic:

$$|f_0 g_0| = |f_0| + |g_0| = |h_0| = 0, \quad |f_1 g_0| = |h_1| = 1, \quad |f_1 g_1| = |h_0| = 0 \tag{2.5}$$

The grade of a differentiation operator depends on the bosonic/fermionic variable i.e :

$$|\frac{\partial g_0}{\partial f_0}| = |\partial/\partial f_0| + |g_0| = 0, \quad |\frac{\partial g_0}{\partial f_1}| = |\partial/\partial f_1| + |g_0| = 1$$

$$|\frac{\partial g_1}{\partial f_0}| = |\partial/\partial f_0| + |g_1| = 1, \quad |\frac{\partial g_1}{\partial f_1}| = |\partial/\partial f_1| + |g_1| = 0 \tag{2.6}$$

where $|\partial/\partial f_0| = 0$ while $|\partial/\partial f_1| = 1$. The grading of the sum of bosons $|f(x)| = 0$ where $f(x) = \sum_{n=0}^{\infty} \frac{f_n}{n!} x^n$. However; in the case of a linear combination of bosonic and fermionic variables

$$F(x,g) = f_0(x) + \sum_{j=1}^{n} \sum_{i=1}^{n} f_0^{ij} g_{ij} \tag{2.7}$$

where f_0^{ij}, f_0 are bosons and g_{ij}'s are fermions, $|F(x,g)|$ is not well defined.

3. GRASSMANN ALGEBRA

In this chapter we will follow [4-6]. Fermions are particles that require anti-commuting numbers called the Grassmann numbers. Their algebra is called Grassmann Algebra. A $(n+1)$ dimensional Grassmann algebra G on $\mathbb{R}$ or $\mathbb{C}$ is formed by a unit 1 and generators θ_i with $i = 1, \ldots, n$ that satisfy the anti-commutation relation

$$\{\theta_i, \theta_j\} = \theta_i\theta_j + \theta_j\theta_i = 0 \quad \forall i, j. \tag{3.1}$$

It follows that in particular the square of any generator vanishes

$$\theta_i^2 = 0 \tag{3.2}$$

by suggesting the Pauli exclusion principle that one cannot put two identical fermions in the same quantum state. Another consequence is :

$$\theta_{i_1}\theta_{i_2}\theta_{i_3}\ldots\theta_{i_n} = \varepsilon_{i_1 i_2 \ldots i_n}\theta_1\theta_2\ldots\theta_n \tag{3.3}$$

$$\theta_{i_1}\theta_{i_2}\ldots\theta_{i_1}\theta_{i_m} = 0 \quad (m > n), \tag{3.4}$$

where

$$\varepsilon_{i_1 i_2 \ldots i_n} = \begin{cases} +1 & \text{if } i_1 \ldots i_n \text{ is an even permutation of } 1 \ldots n \\ -1 & \text{if } i_1 \ldots i_n \text{ is an odd permutation of } 1 \ldots n \\ 0 & \text{otherwise} \end{cases}$$

An arbitrary element f of the Grassmann algebra can be expanded as

$$f(\theta) = f_0 + \sum_{i=1}^{n} f_i\theta_i + \sum_{i<j} f_{ij}\theta_i\theta_j + \ldots$$

$$= \sum_{0\leq k\leq n} \frac{1}{k!} \sum_{i_1\ldots i_k} f_{i_1,\ldots i_k}\theta_{i_1\ldots i_k}, \tag{3.5}$$

where the coefficients $f_0, f_i, f_{ij}, \ldots$ and $f_{i_1,\ldots i_k}$ are real or complex numbers and do not depend on θ_i's. They are antisymmetric under the exchange of any two indices. The elements of a Grassmann algebra with only one or two generators are

$$f(\theta) = f_0 + f_1\theta \tag{3.6}$$

and similarly for two generators

$$f(\theta_1, \theta_2) = f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2. \tag{3.7}$$

We don't take higher order terms in the expansion because the square of generators needs to be zero. We can separate the Grassmann Algebra into two subspaces,

$$G = G_0 + G_1. \tag{3.8}$$

It forms $\mathbb{Z}_2$ Graded algebras in chapter 2 with the even and odd number of generators which are elements of the subspaces G_0 and G_1,respectively. Terms with an even number of θ's are called Grassmann even or bosons. Their grade is set to be 0. They commute with each element of the Grassmann algebra while the odd number of generators are called Grassmann odd or fermions with the grade one. They satisfy the anti-commutation property among each other.

3.1. Differentiation

In Grassmann algebra it is possible to use two types of differentiation i.e left and right. For the function (3.7), the derivation depends on whether the variable is on the left or right of its Taylor expansion. Left derivatives are defined by removing the variable to the left side

$$\frac{\partial}{\partial\theta_1}\,|_L\, f(\theta_1,\theta_2) = f_1 + f_{12}\theta_2 \tag{3.9}$$

Similarly, right derivatives are obtained by removing the variable to the right

$$\frac{\partial}{\partial\theta_1}\,|_R\, f(\theta_1,\theta_2) = f_1 - f_{12}\theta_2 \tag{3.10}$$

where a minus sign emerges because of the anti-commutation between θ_1 and θ_2. The subscripts L and R on the partial derivatives show us in which direction the function is differentiated. The relation between the right and left derivation is

$$\frac{\partial}{\partial\theta_i}\,|_R\, \theta_{i_1}\ldots\theta_{i_k} = (-1)^{k-1}\frac{\partial}{\partial\theta_i}\,|_L\, \theta_{i_1}\ldots\theta_{i_k}. \tag{3.11}$$

For an odd number of generators both type of derivations are the same but for an even number they differ by a sign change [7]. In this thesis, the right derivation ,which is shifting θ_i to the right of the monomial, is used. We will not use the $|_R$ with $\partial/\partial\theta_i$ as a notation for the right differentiation in the next chapters. The differential operator acts on a generator from the right and results in a Kronecker delta. The Leibniz rule takes the form:

$$\begin{aligned}\frac{\partial}{\partial\theta_i}(\theta_j\theta_k) &= \theta_j\frac{\partial\theta_k}{\partial\theta_i} - \theta_k\frac{\partial\theta_j}{\partial\theta_i}\\ &= \theta_j\delta_{ik} - \theta_k\delta_{ij}.\end{aligned} \tag{3.12}$$

It can be generalized as

$$\frac{\partial}{\partial\theta_i}(\theta_{i_1}\ldots\theta_{i_k}) = (\theta_{i_1}\ldots\theta_{i_{k-1}})\delta_{ii_k} - (\theta_{i_1}\ldots\theta_{i_{k-2}}\theta_{i_k})\delta_{ii_{k-1}} + \ldots + (-1)^{k-1}(\theta_{i_2}\ldots\theta_{i_k})\delta_{ii_1}. \tag{3.13}$$

The chain rule in derivation with respect to Grassmann variables is different from the usual one; the order of the terms is important. Consider two functions f and g similar to the function (3.6) where $f_1 = \partial f/\partial\theta$ and $g_1 = \partial g/\partial\theta$. We can write f in terms of g

:

$$(f \circ g)(\theta) = f(g(\theta)) \tag{3.14}$$

$$\begin{aligned} &= f(g_0) + \frac{\partial f}{\partial g} g_1 \theta & (3.15) \\ &= f(g_0) + \frac{\partial f}{\partial g}\frac{\partial g}{\partial \theta}\theta. & (3.16) \end{aligned}$$

(3.14) is rewritten in another form:

$$(f \circ g)_0 + (f \circ g)_1 \theta = (f \circ g)_0 + \frac{\partial}{\partial \theta}(f \circ g)\theta. \tag{3.17}$$

When (3.16) and (3.17) are matched , one notes that the chain rule is

$$\frac{\partial}{\partial \theta}(f \circ g) = \frac{\partial f}{\partial g}\frac{\partial g}{\partial \theta}. \tag{3.18}$$

Here, the function is firstly differentiated with respect to g and then with respect to θ. The differentiation operators $\partial/\partial\theta_i$ and θ_i's satisfy the anti-commutation property. In [4], the anti-commutation relations are given as

$$\theta_i\theta_j + \theta_j\theta_i = 0, \quad \frac{\partial}{\partial \theta_i}\frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial \theta_j}\frac{\partial}{\partial \theta_i} = 0, \quad \theta_i \frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial \theta_i}\theta_j = \delta_{ij}. \tag{3.19}$$

3.2. Integration

The integration over Grassmann variables is identical to differentiation. It can be defined as

$$\int d\theta f(\theta) \equiv \frac{\partial}{\partial \theta} f(\theta). \tag{3.20}$$

We can write it in this form because the integration of a total derivative gives the surface term and it is zero

$$\int d\theta \frac{\partial}{\partial\theta} f(\theta) = f(\theta_f) - f(\theta_i) = 0. \tag{3.21}$$

Also, a derivative of a definite integral is zero $\partial/\partial\theta \int d\theta f(\theta) = 0$. Clearly, the right hand side of (3.21) and (3.2) vanishes with the square of the differentiation operator. These three properties require the integration and differentiation are similar. If there are n generators ,

$$\int d\theta_1 d\theta_2 \ldots d\theta_n f(\theta_1, \theta_2, \ldots \theta_n) = \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2} \cdots \frac{\partial}{\partial\theta_n} f(\theta_1, \theta_2, \ldots \theta_n). \tag{3.22}$$

Some examples are given

$$\int d\theta = \frac{\partial}{\partial\theta} 1 = 0, \qquad \int \theta d\theta = \frac{\partial\theta}{\partial\theta} = 1. \tag{3.23}$$

We can generalize it as follows

$$\int \theta_1 \ldots \theta_n d\theta_1 \ldots d\theta_n = 1. \tag{3.24}$$

3.3. Complex Conjugation

In Grassmann algebra we can define a complex conjugation similar to the Hermitian conjugation for matrices and operators. Hermitian conjugation for Grassmann generators is given as

$$\theta^\dagger = \overline{\theta}, \quad \overline{\theta}^\dagger = \theta. \tag{3.25}$$

In the case of two generators, we change the order and take the complex conjugate of each term. Some examples are given in [4] :

$$(\theta\overline{\theta})^\dagger = \theta\overline{\theta} \tag{3.26}$$

$$(\lambda A_1 + \mu A_2)^\dagger = \bar{\lambda} A_1^\dagger + \bar{\mu} A_2^\dagger \tag{3.27}$$

$$(A_1 A_2)^\dagger = A_2^\dagger A_1^\dagger \tag{3.28}$$

where $\forall A_1, A_2 \in G \quad \lambda, \mu \in \mathbb{C}$.

4. CLASSICAL MECHANICS WITH FERMIONS AND BOSONS

In this chapter, we review some general results with the addition of fermions to a bosonic system in [7-10]. In some books the mechanics describing such a mixed system is called Pseudo classical mechanics. Now, consider a classical system with n bosonic degrees of freedom $q^i = (q_1 \dots q_n)$ and k fermionic degrees of freedom $\psi^i = (\psi_1 \dots \psi_k)$. The Lagrangian depends on q,ψ and t:

$$L = L(q^i, \dot{q}^i, \psi^i, \dot{\psi}^i, t). \tag{4.1}$$

The condition for the stationary action is

$$\delta S(\psi) = \delta \int_{t_1}^{t_2} L(q, \dot{q}, \psi, \dot{\psi}, t) dt = 0, \quad \delta\psi(t_1) = \delta\psi(t_2) = 0, \quad \delta q(t_1) = \delta q(t_2) = 0 \tag{4.2}$$

where the Lagrangian L is an even Grassmann function. The variational derivatives with respect to Grassmann variables obey the Leibniz rule. In particular,

$$\frac{\delta}{\delta\psi^i(t)}\psi^j(t') = \delta_{ij}\delta(t - t') \tag{4.3}$$

So one finds

$$\delta S(\psi) = \int_{t_1}^{t_2} dt (\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i + \frac{\partial L}{\partial \dot{\psi}^i}\delta \dot{\psi}^i + \frac{\partial L}{\partial \psi^i}\delta \psi^i) = 0. \tag{4.4}$$

Since the variations $\delta\psi^i$ anti-commute with ψ^i, one must be careful with the ordering in the products in the right-hand side. After integration by parts,

$$= \int_{t_1}^{t_2} dt \left((-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial q^i})\delta q^i + (-\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}^i} + \frac{\partial L}{\partial \psi^i})\delta\psi^i \right) = 0.$$

Due to the arbitrary $\delta\psi^i$ and δq^i the equations of motion are found as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} \tag{4.5}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}^i} = \frac{\partial L}{\partial \psi^i}. \tag{4.6}$$

The canonical momentum conjugated to q^i and ψ^i are found by taking the derivative of the Lagrangian with respect to $\dot{q}^i$ and $\dot{\psi}^i$

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \qquad \pi_i = \frac{\partial L}{\partial \dot{\psi}^i}. \tag{4.7}$$

The canonical momenta are defined as Grassmann even and odd functions since the grades of momenta are $|p_i| = 0$ and $|\pi_i| = |\partial/\partial\dot{\psi}^i| + |L| = 1$. The Hamiltonian is found by using the Legendre transform of the Lagrangian:

$$H = p_i\dot{q}^i + \pi_i\dot{\psi}^i - L. \tag{4.8}$$

It is easy to see the Hamiltonian is an even function of the Grassmann algebra by checking its grade. Since the canonical momenta and velocities for fermions are odd and anti-commute with each other, the order of the product of π_i and $\dot{\psi}^i$ is relevant. When we change $\pi_i\dot{\psi}^i$ with $\dot{\psi}^i\pi_i = -\pi_i\dot{\psi}^i$,sign changes which means, we replace the left derivatives by the right derivatives. The Hamiltonian equations of motion are derived from the stationary action by writing the Lagrangian in terms of the Hamiltonian form [15],

$$\delta S(\pi, \psi) = \delta \int_{t_1}^{t_2} Ldt = \delta \int_{t_1}^{t_2} dt(p_i\dot{q}^i + \pi_i\dot{\psi}^i - H) = 0, \quad \delta q^i(t_a) = \delta\psi^i(t_a) = 0, \quad a = 1, 2 \tag{4.9}$$

Applying the variation inside the integral, one finds

$$= \int_{t_1}^{t_2} dt \left((\dot{q}^i\delta p_i - \dot{p}_i\delta q^i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i) + (-\dot{\psi}^i\delta\pi_i - \dot{\pi}_i\delta\psi^i - \frac{\partial H}{\partial \psi^i}\delta\psi^i - \frac{\partial H}{\partial \pi_i}\delta\pi_i) \right)$$

$$\int_{t_1}^{t_2} dt \left((\dot{q}^i - \frac{\partial H}{\partial p_i})\delta p_i - (\dot{p}_i + \frac{\partial H}{\partial q_i})\delta q^i + (-\dot{\psi}^i - \frac{\partial H}{\partial \pi_i})\delta \pi_i + (-\dot{\pi}_i - \frac{\partial H}{\partial \psi_i})\delta \psi^i \right) = 0 \tag{4.10}$$

where the variations $\delta\pi_i$ and $\delta\psi^i$ are Grassmann odd and anti-commute with π_i and ψ^i. The term in the parenthesis must be zero. Because $\delta\pi_i$ and $\delta\psi^i$ are arbitrary, the following equations of motion should be satisfied

$$\dot{q}^i = \frac{\partial H}{\partial p_i}; \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{4.11}$$

$$\dot{\psi}^i = -\frac{\partial H}{\partial \pi_i}; \qquad \dot{\pi}_i = -\frac{\partial H}{\partial \psi_i}. \tag{4.12}$$

Consider a function $F = F(q, p, \pi, \psi, t)$ on the phase space. Its time evolution along the trajectory of motion is defined by the Hamiltonian equations of motion:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial p_i}\dot{p}^i + \frac{\partial F}{\partial \psi^i}\dot{\psi}^i + \frac{\partial F}{\partial \pi_i}\dot{\pi}_i \tag{4.13}$$

$$= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} - (\frac{\partial F}{\partial \psi^i}\frac{\partial H}{\partial \pi_i} + \frac{\partial F}{\partial \pi_i}\frac{\partial H}{\partial \psi^i}) \equiv \{F, H\}. \tag{4.14}$$

If F does not explicitly depend on time, then $dF/dt = \{F, H\}$. After replacing F with the canonical variables, the Hamiltonian equations of motion can be written in terms of Poisson brackets:

$$\dot{q}^i = \{q^i, H\}, \quad \dot{p}_i = \{p_i, H\} \tag{4.15}$$

$$\dot{\psi}^i = -\{\psi^i, H\}, \quad \dot{\pi}_i = -\{\pi^i, H\}. \tag{4.16}$$

More generally, the generalized graded Poisson bracket is defined as follows:

$$\{f, g\} = \frac{\partial f}{\partial \Theta_i}\frac{\partial g}{\partial P_i} - (-1)^{|g||f|}\frac{\partial g}{\partial \Theta_i}\frac{\partial f}{\partial P_i}. \tag{4.17}$$

where $\Theta^i = (q^i, \psi^i)$ and $P_i = (p_i, \pi_i)$. The grade of the Poisson bracket is the sum of grades of two functions

$$|\{f, g\}| = |f| + |g|. \tag{4.18}$$

In the case of both bosonic and fermionic variables, it is

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - (-1)^{|g||f|}\frac{\partial g}{\partial q}\frac{\partial f}{\partial p} + \frac{\partial f}{\partial \psi}\frac{\partial g}{\partial \pi_\psi} - (-1)^{|g||f|}\frac{\partial g}{\partial \psi}\frac{\partial f}{\partial \pi_\psi}. \tag{4.19}$$

For various cases Poisson bracket can be summarized similar to [10] :

$$\begin{aligned}
\{B_1,B_2\} &= \frac{\partial B_1}{\partial q}\frac{\partial B_2}{\partial p} - \frac{\partial B_2}{\partial q}\frac{\partial B_1}{\partial p} + \frac{\partial B_1}{\partial \psi}\frac{\partial B_2}{\partial \pi_\psi} - \frac{\partial B_2}{\partial \psi}\frac{\partial B_1}{\partial \pi_\psi} && (4.20)\\
\{B,F\} &= \frac{\partial B}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial B}{\partial p} + \frac{\partial B}{\partial \psi}\frac{\partial F}{\partial \pi_\psi} - \frac{\partial F}{\partial \psi}\frac{\partial B}{\partial \pi_\psi} && (4.21)\\
\{F,B\} &= \frac{\partial F}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial q}\frac{\partial F}{\partial p} + \frac{\partial F}{\partial \psi}\frac{\partial B}{\partial \pi_\psi} - \frac{\partial B}{\partial \psi}\frac{\partial F}{\partial \pi_\psi} && (4.22)\\
\{F_1,F_2\} &= \frac{\partial F_1}{\partial q}\frac{\partial F_2}{\partial p} + \frac{\partial F_2}{\partial q}\frac{\partial F_1}{\partial p} + \frac{\partial F_1}{\partial \psi}\frac{\partial F_2}{\partial \pi_\psi} + \frac{\partial F_2}{\partial \psi}\frac{\partial F_1}{\partial \pi_\psi}. && (4.23)
\end{aligned}$$

The graded Poisson bracket comes from the product definition in $\mathbb{Z}_2$ graded algebra so obeys the similar properties in Chapter 2. It has the following permutation property:

$$\{f,g\} = -(-1)^{|f||g|}\{g,f\} \tag{4.24}$$

which is known as the Super Skew Symmetry. As an interpretation, the Poisson bracket of two bosons or fermion and boson is antisymmetric. However, Poisson bracket of two fermions doesn't obey this rule; it is symmetric. The property (4.24) doesn't hold with the Poisson bracket of the linear combinations of mixed elements in the Grassmann algebra. The permutation rule for them is based on the bi-linearity of the Poisson bracket. Poisson brackets satisfy the following properties:

(i) Graded Leibniz rule:

$$\{f,gh\} = \{f,g\}h + (-1)^{|f||g|}g\{f,h\} = \{f,g\}h + (-1)^{|g||h|}\{f,h\}g$$

(ii) Bi-linearity:

$$\{\alpha f_1 + \beta g_1, \gamma f_2 + \delta g_2\} = \alpha\gamma\{f_1,f_2\} + \alpha\delta\{f_1,g_2\} + \beta\gamma\{g_1,f_2\} + \beta\delta\{g_1,g_2\}$$

(iii) Super Jacobi Identity:

$$(-1)^{|f||h|}\{\{f,g\},h\}+(-1)^{|g||f|}\{\{g,h\},f\}+(-1)^{|h||g|}\{\{h,f\},g\}=0$$

4.1. Noether Theorem

Noether theorem states that if the Lagrangian of a system has a continuous symmetry, there exists an associated conserved quantity [11], [12]. Symmetry means that Lagrangian is unaffected by any transformation of the generalized coordinates q, velocities $\dot{q}$ and time t. Continuous symmetry requires a continuous constant parameter denoted by an infinitesimal ϵ.

4.1.1. Examples for Noether Theorem

If q is a cyclic variable, the associated conjugate momentum is conserved.

$$\frac{\partial L}{\partial q}=0, \qquad p_i=\frac{\partial L}{\partial \dot{q}}=0 \tag{4.25}$$

which is the simplest example of Noether's theorem [11]. In the case of the cyclic coordinate discussed above, the corresponding symmetry is simply

$$q(t)\to q(t)+\epsilon; \qquad \dot{q}(t)\to\dot{q}(t); \qquad t\to t \tag{4.26}$$

The Lagrangian changes at first order in ϵ as follows

$$\delta L\equiv L(q+\epsilon,\dot{q};t)-L(q,\dot{q};t)\simeq\frac{\partial L}{\partial q}\epsilon, \tag{4.27}$$

which vanishes if an only if q is cyclic. Consider a Lagrangian system with n degrees of freedom $q_1 ... q_n$. Transformation of general coordinates for constant infinitesimal ϵ

and for functions $\gamma_i(t)$, when time is unchanged,

$$\begin{aligned} q_i(t) &\rightarrow q_i(t) + \epsilon\gamma_i(t) \\ \dot{q}_i(t) &\rightarrow \dot{q}_i(t) + \epsilon\dot{\gamma}_i(t) \\ t &\rightarrow t \end{aligned} \tag{4.28}$$

is a symmetry. Then it doesn't affect the Lagrangian

$$\delta L \equiv \epsilon \sum_i (\frac{\partial L}{\partial q_i}\gamma_i + \frac{\partial L}{\partial \dot{q}_i}\dot{\gamma}_i) = 0 \tag{4.29}$$

We plug Euler Lagrange equations into the variation. It gives us a total time derivation:

$$\epsilon \frac{d}{dt}(\sum_i \frac{\partial L}{\partial \dot{q}_i}\gamma_i) = 0. \tag{4.30}$$

Then the quantity

$$\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\gamma_i \tag{4.31}$$

is a constant of motion or conserved quantity.

4.1.1.1. Time Translation. We shift time by an infinitesimal constant ϵ.

$$q_i(t) \rightarrow q_i(t), \quad \dot{q}_i(t) \rightarrow \dot{q}_i(t), \quad t \rightarrow t + \epsilon \tag{4.32}$$

This is a symmetry if and only if the Lagrangian does not depend explicitly on time. When we apply the transformations

$$L(q, \dot{q}; t + \epsilon) \;=\; L(q, \dot{q}; t) + \frac{\partial L}{\partial t}\epsilon, \tag{4.33}$$

Lagrangian changes by a factor of $(\partial L/\partial t)\epsilon$ which vanishes if the partial time-derivative of the Lagrangian is zero. The total time-derivation of the Lagrangian is

$$\begin{aligned} \frac{d}{dt}L &= \frac{\partial L}{\partial t} + \sum_i (\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i) \\ &= \frac{\partial L}{\partial t} + \frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i. \end{aligned} \tag{4.34}$$

We are left with the relation $dH/dt = -\partial L/\partial t$. The Hamiltonian is

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L. \tag{4.35}$$

If the Lagrangian doesn't change under time translations, which means it does not depend on time explicitly, then the Hamiltonian of the system is conserved. Hamiltonian corresponds to total energy in time independent systems. Therefore, the invariance of the Lagrangian under time translations yields the conservation of energy.

4.1.2. Noether Theorem with Fermions and Bosons

Noether theorem with fermionic and bosonic degrees of freedom can be derived by using the transformation properties of the Lagrangian $L = L(\dot{n}, n, t)$ where n^a's denote q^i and ψ^i. We can find a trajectory of the system in the super-space generated by bosons and fermions if they are time dependent $n^a = n^a(t)$. We transform the trajectory infinitesimally

$$n^a_\epsilon(t) = n^a(t) + \epsilon\Theta^a(\dot{n}, n, t) + O(\epsilon^2) \tag{4.36}$$

where the transformation parameter ϵ is either a real or Grassmann variable and $\Theta^a = (\partial n^a_\epsilon/\partial\epsilon)_\epsilon = 0$. If ϵ is a Grassmann variable, the transformation is up to ϵ^2 order because it is zero. When trajectory is transformed, grades must be also conserved. Therefore, if the grade of ϵ is given, we can fix the grade of the functions Θ uniquely.

Assume that the Lagrangian has the following transformation

$$L'_\epsilon(\dot{n}_\epsilon, n_\epsilon, t) = L(\dot{n}_\epsilon, n_\epsilon, t) - \epsilon \frac{d}{dt} F(\dot{n}, n, t) + O(\epsilon^2) \tag{4.37}$$

where

$$L(\dot{n}_\epsilon, n_\epsilon, t) = L(\dot{n}, n, t) + \epsilon \frac{\partial L}{\partial n} \Theta + \epsilon \frac{\partial L}{\partial \dot{n}} \dot{\Theta} + O(\epsilon^2). \tag{4.38}$$

We use $d/dt(\partial L/\partial \dot{n}) = \partial L/\partial n$ in the expansion of the Lagrangian and obtain the variation

$$\epsilon \frac{d}{dt}(\Theta \frac{\partial L}{\partial \dot{n}} - F) = 0 \tag{4.39}$$

Then the quantity

$$Q = \Theta^a \frac{\partial L}{\partial \dot{n}^a} - F \tag{4.40}$$

is an integral of motion. Calculating the variation of the Lagrangian under the trajectory variation one finds

$$\frac{dF}{dt} = \dot{\Theta}^a \frac{\partial L}{\partial \dot{n}^a} + \Theta^a \frac{\partial L}{\partial n^a} \tag{4.41}$$

Differentiation of the integral of motion is

$$\frac{dQ}{dt} = \dot{\Theta}^a \frac{\partial L}{\partial \dot{n}^a} + \Theta^a \frac{d}{dt} \frac{\partial L}{\partial \dot{n}^a} - \frac{dF}{dt} = \dot{\Theta}^a \frac{\partial L}{\partial \dot{n}^a} + \Theta^a \frac{\partial L}{\partial n^a} - \frac{dF}{dt} = 0 \tag{4.42}$$

If the action is invariant under the transformation Θ^a and a time transformation

$$t \to t_\epsilon = t + \epsilon T \qquad \text{with} \qquad T = (\partial t_\epsilon / \partial \epsilon)_{\epsilon=0}. \tag{4.43}$$

For the corresponding velocity transformation $\delta\dot{n}$

$$\dot{n}^a_\epsilon = \frac{dn^a_\epsilon}{dt_\epsilon} = \frac{dn^a + \epsilon d\Theta^a}{dt + \epsilon dT} = \frac{\dot{n}^a + \epsilon\dot{\Theta}^a}{1 + \epsilon\dot{T}} = \dot{n}^a + \epsilon\dot{\Theta}^a - \epsilon\dot{T}\dot{n}^a + O(\epsilon^2) \tag{4.44}$$

We get

$$\delta\dot{n} = \epsilon(\dot{\Theta} - \dot{T}\dot{n}). \tag{4.45}$$

Now differently from the previous calculations we change the velocity transformation and rewrite the Taylor expansion of $L(\dot{n}^a_\epsilon, n^a_\epsilon, t_\epsilon)$

$$L(\dot{n}_\epsilon, n_\epsilon, t_\epsilon) = L(\dot{n}, n, t) + \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial n}\delta n + \frac{\partial L}{\partial \dot{n}}\delta\dot{n} + \dots \tag{4.46}$$

Inserting δt,δn and $\delta\dot{n}$ we obtain an equation for F

$$\frac{dF}{dt} - \dot{T}L(n^a, \dot{n}^a, t) - \frac{\partial L}{\partial t}T - \frac{\partial L}{\partial n^a}\Theta^a - \frac{\partial L}{\partial \dot{n}^a}(\dot{\Theta}^a - \dot{T}\dot{n}^a). \tag{4.47}$$

The terms can be split into a total time derivative and a sum containing the Euler-Lagrange equations

$$\frac{d}{dt}(F(n^a, t) - TL + (T\dot{n}^a - \Theta^a)\frac{\partial L}{\partial \dot{n}^a}) + (T\dot{n}^a - \Theta^a)(\frac{\partial L}{\partial n^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{n}^a}) = 0. \tag{4.48}$$

The expression in the parenthesis yields a conserved quantity Q along the solution $n^a, \dot{n}^a$ of the Euler-Lagrange equations

$$Q = F(n^a, t) - TL + (T\dot{n}^a - \Theta^a)\frac{\partial L}{\partial \dot{n}^a} \Leftrightarrow \frac{\partial L}{\partial n^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{n}^a} = 0. \tag{4.49}$$

Legendre transformation relates the Lagrangian with the corresponding Hamiltonian

$$L(n^a, \dot{n}^a, t) = \frac{\partial L}{\partial \dot{n}^a}\dot{n}^a - H(n^a, \dot{n}^a, t), \quad , \frac{\partial L}{\partial t} = -\frac{dH}{dt} \tag{4.50}$$

Applying these rules to the Noether invariant, we find

$$Q = TH(n^a, \dot{n}^a, t) - \Theta^a \frac{\partial L}{\partial \dot{n}^a} + F(n^a, t). \tag{4.51}$$

For $F = 0$ the Noether integral of motion is

$$Q = T(\frac{\partial L}{\partial \dot{n}^a}\dot{n}^a - L) - \Theta^a \frac{\partial L}{\partial \dot{n}^a}. \tag{4.52}$$

4.2. Lagrangian with One Complex Fermion

As an example to a classical system which has only one complex fermionic degree of freedom, consider the Lagrangian describing the system is

$$L = \frac{i}{2}(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}) - V_1\bar{\psi}\psi \tag{4.53}$$

where V_1 is a bosonic function. First we should check the reality condition on the Lagrangian by taking the hermitian conjugation. We change the order of the product and then take the complex conjugate:

$$\begin{aligned} L^\dagger &= (\frac{i}{2}(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}) - V_1\bar{\psi}\psi)^\dagger \\ &= -\frac{i}{2}(\psi^\dagger\dot{\bar{\psi}}^\dagger - \dot{\psi}^\dagger\bar{\psi}^\dagger) - V_1^\dagger\psi^\dagger\bar{\psi}^\dagger \\ &= -\frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - V_1\bar{\psi}\psi \end{aligned} \tag{4.54}$$

which is L itself. The equation of motion for the Lagrangian :

$$\dot{\psi} - iV_1\psi = 0 \qquad \Rightarrow \psi = e^{iV_1 t}\psi_0 \tag{4.55}$$

where ψ_0 is the initial fermion. The momentum is

$$\pi = -\frac{i}{2}\bar{\psi}. \tag{4.56}$$

Plugging them into (4.8), we obtain

$$\begin{aligned} H &= -\frac{i}{2}\bar{\psi}\dot{\psi} - \frac{i}{2}\psi\dot{\bar{\psi}} - \frac{i}{2}(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}) + V_1\bar{\psi}\psi \\ &= V_1\psi\bar{\psi} \qquad (4.57) \\ &= V_1\psi_0\bar{\psi}_0 \qquad (4.58) \end{aligned}$$

Since ψ_0 is a constant, Hamiltonian is a constant. Again, this result proves that Hamiltonian is really the conserved quantity of the system.

4.3. Lagrangian with One Complex Fermion and a Boson

The simplest non trivial example of a classical system with bosonic and fermionic degree of freedom is described by a Lagrangian given in [13],[14].

$$L = \frac{1}{2}\dot{x}^2 - V_1(x) + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - V_2(x)\bar{\psi}\psi \qquad (4.59)$$

where $V_1(x)$ is a bosonic potential and $V_2(x)$ describes a coupling between the boson and the fermion. The equation of motion for x

$$\ddot{x} + V_1'(x) + V_2'(x)\bar{\psi}\psi = 0. \qquad (4.60)$$

Equations of motion for fermion are

$$\dot{\bar{\psi}} - iV_2(x)\bar{\psi} = 0 \quad \text{and} \quad \dot{\psi} + iV_2(x)\psi = 0. \qquad (4.61)$$

With the initial conditions $\psi(0) = \psi_0$, $\bar{\psi}(0) = \bar{\psi}_0$ fermionic solutions are

$$\bar{\psi}(t) = \bar{\psi}_0 e^{i\int_0^t dt' V_2(x(t'))} \quad \text{and} \quad \psi(t) = \psi_0 e^{-i\int_0^t dt' V_2(x(t'))}. \qquad (4.62)$$

One notes that the multiplication of fermion with its complex conjugate is a constant which is $\bar{\psi(t)}\psi(t) = \bar{\psi}_0\psi_0$. So (4.60) can be rewritten as

$$\ddot{x} = -V_1'(x) - V_2'(x)\bar{\psi}_0\psi_0. \tag{4.63}$$

We look for a solution like

$$x(t) = x_{qc}(t) + q(t)\bar{\psi}_0\psi_0 \tag{4.64}$$

where $x_{qc}(t)$ is the quasi-classical solution that describes the bosonic classical dynamics along with fermionic degrees of freedom[14] and $x_{qc}(t), q(t)$ are real functions. The quasi-classical expansion tells us that the classical solution always involves Grassmann terms with the quasi-classical solution. It is a Grassmann valued function because of the coupling of boson and fermion. We cut the expansion at the first order of $\bar{\psi}_0\psi_0$ term because the second and higher order terms are zero due to the anti-commutation of fermions. The classical solution $x(t)$ and the quasi-classical solution $x_{qc}(t)$ are equal only for the special initial condition $\psi(0) = \bar{\psi}(0) = 0$. For the potential quasi-classical expansion is

$$V(x) = V(x_{qc}) + V'(x_{qc})q(t)\bar{\psi}_0\psi_0. \tag{4.65}$$

The equations of motion become

$$\ddot{x}_{qc}(t) + V_1'(x_{qc}) + (\ddot{q} + V_1''(x_{qc})q + V_2'(x_{qc}))\bar{\psi}_0\psi_0 = 0 \tag{4.66}$$

and

$$\dot{\psi} + iV_2(x_{qc})\psi = 0. \tag{4.67}$$

We obtain fermion equations in terms of quasi-classical approach as

$$\psi(t) = \psi_0 e^{-i\int_0^t dt' V_2(x_{qc}(t'))}, \quad \bar{\psi}(t) = \bar{\psi}_0 e^{i\int_0^t dt' V_2(x_{qc}(t'))} \tag{4.68}$$

The Hamiltonian of the system is

$$H = \frac{1}{2}\dot{x}^2 + V_1(x) + V_2(x)\bar{\psi}_0\psi_0. \tag{4.69}$$

We can separate it to both bosonic and fermionic part; so the quasi-classical expansion of the Hamiltonian can be written as

$$H = E + F\bar{\psi}_0\psi_0. \tag{4.70}$$

When we put (4.64) in the Hamiltonian, we get

$$H = \frac{1}{2}(\dot{x}_{qc}(t) + \dot{q}\bar{\psi}_0\psi_0)^2 + V_1(x_{qc}) + V_1'(x_{qc})q\bar{\psi}_0\psi_0 + V_2(x_{qc})\bar{\psi}_0\psi_0. \tag{4.71}$$

After eliminating some terms we are left with the final result

$$H = \frac{1}{2}\dot{x}_{qc}^2(t) + V_1(x_{qc}) + (\dot{q}\dot{x}_{qc}(t) + V_1'(x_{qc})q + V_2(x_{qc}))\bar{\psi}_0\psi_0 \tag{4.72}$$

where

$$E = \frac{1}{2}\dot{x}_{qc}^2(t) + V_1(x_{qc}) \quad \text{and} \quad F = \dot{q}\dot{x}_{qc}(t) + V_1'(x_{qc})q + V_2(x_{qc}). \tag{4.73}$$

E is the usual energy along the quasi-classical path. We can find a solution for $q(t)$ in terms of the quasi-classical solution

$$\dot{q} = \frac{1}{\dot{x}_{qc}(t)}(F - V_1'(x_{qc})q - V_2(x_{qc})) \tag{4.74}$$

From the energy conservation we have $\dot{x}_{qc}(t) = \sqrt{2(E - V_1)}$ which vanishes at the turning points. We can solve q(t) by using $\ddot{x}_{qc} = -V_1'(x_{qc})$. Assume

$$q(t) = \dot{x}_{qc}(t) f(t). \tag{4.75}$$

Time derivative of $q(t)$ is

$$\begin{aligned} \dot{q}(t) &= \ddot{x}_{qc}(t) f(t) + \dot{x}_{qc}(t) \dot{f}(t) \\ &= -V_1'(x_{qc}) \frac{q}{\dot{x}(t)} + \dot{x}_{qc}(t) \dot{f}(t) \end{aligned} \tag{4.76}$$

It should be the same with (4.74) so we get

$$\begin{aligned} \dot{f}(t) &= \frac{F - V_2(t)}{\dot{x}_{qc}^2} \\ &= \frac{F - V_2(x_{qc}(t))}{2(E - V_1(x_{qc}(t)))} \end{aligned} \tag{4.77}$$

Doing integration, $f(t)$ is found

$$f(t) = \int_0^t dt' \frac{F - V_2(x_{qc}(t'))}{2(E - V_1(x_{qc}(t')))} + f_0. \tag{4.78}$$

Exact solution of $q(t)$

$$q(t) = \frac{\dot{x}_{qc}(t)}{\dot{x}_{qc}(0)} (q(0) + \frac{\dot{x}_{qc}(0)}{2} \int_0^t dt' \frac{F - V_2(x_{qc}(t'))}{E - V_1(x_{qc}(t'))}) \tag{4.79}$$

where $q(0)$ is a constant of integration. F and E are arbitrary constants of integration but related to the conservation of energy. One should note that when $q(0)$ is zero, $q(t)$ doesn't vanish due to the integration for $t > 0$. Therefore, even the classical solution is real initially $x(0) \in \mathbb{R}$, it will be a Grassmann valued solution.

4.4. Adiabatic Invariant

Consider an oscillating plane pendulum with a mass. It is pulled up or down very slowly so that the length of the pendulum changes very little in a period of motion. The energy of the system is not constant any more since the string length varies with time. The frequency of the motion changes as well but the ratio E/w remains constant over a long period. Consider a system with a parameter λ which is time independent initially. After a while λ changes so slowly over a long interval time that it behaves like a constant during the motion. When the value of the parameter is constant, the motion becomes periodic; slow changes does not alter the periodic motion. The Hamiltonian of the system is described by the action angle variables (I, θ) and the adiabatic invariant is precisely is the action variable I [15-17]. For the 1-D harmonic oscillator example, the Hamiltonian is

$$H = \frac{p^2}{2m} + V(q, w(t)) \tag{4.80}$$

When the frequency w changes, energy of the system changes as

$$\dot{E} = \frac{\partial H}{\partial \lambda}\dot{w}. \tag{4.81}$$

Assume the adiabatic invariant is

$$I = \frac{1}{2\pi}\oint pdq. \tag{4.82}$$

where the path configuration in phase space is time dependent and given by

$$p = \sqrt{2mE(t) - m^2 w(t)^2 q^2}. \tag{4.83}$$

The action is written explicitly

$$I = \frac{1}{2\pi}\oint_E \sqrt{2m(E(t) - \frac{1}{2}mw(t)^2q^2))}dq \quad (4.84)$$

$$= \frac{1}{2\pi}mw\int_{-q}^{q}\sqrt{A^2 - q^2}dq \quad (4.85)$$

where $A^2 = 2mE/m^2w^2$. The adiabatic invariant becomes

$$I = \frac{E}{w}. \quad (4.86)$$

If we change w slowly, the average of the ratio E/w gives a constant value over a long period. I is the function of $E(t)$ and $w(t)$. I changes with time as follows:

$$\dot{I} = \frac{\partial I}{\partial E}\dot{E} + \frac{\partial I}{\partial w}\dot{w} \quad (4.87)$$

We know the relation between E and w (4.81). On the right hand side terms cancel each other approximately and we get a constant adiabatic invariant [17]. The average of $\dot{I}$ over a period is zero.

$$< \dot{I} >= 0. \quad (4.88)$$

Adiabatic invariant for a fermionic system is similar to the harmonic oscillator case. The Lagrangian is

$$L_\psi = i\bar{\psi}\dot{\psi} - \bar{\psi}\mathbf{x}.\boldsymbol{\sigma}\psi \quad (4.89)$$

with ψ is fermion which is vector of Grassmann variables. The energy of the system is computed as

$$E = \bar{\psi}\mathbf{x}.\boldsymbol{\sigma}\psi. \quad (4.90)$$

This Lagrangian will be studied in detail when we discuss the toy model later. Solutions for fermions are

$$\psi = e^{-i\mathbf{x}.\boldsymbol{\sigma}t}\psi_0. \tag{4.91}$$

They oscillate with a frequency x. Action is the adiabatic invariant.

$$\begin{aligned} I &= \frac{1}{2\pi}\oint_{E=cons.} \pi_\psi d\psi && (4.92) \\ &= \frac{i}{2\pi}\int_0^{\frac{2\pi}{x}} \bar{\psi}\dot{\psi}dt && (4.93) \end{aligned}$$

$$= \frac{1}{2\pi}\int_0^{\frac{2\pi}{x}} \bar{\psi}x_i\sigma_i\psi dt \tag{4.94}$$

We immediately see the term inside the integration is the energy. Since it is constant over a long period, we can take it out of the integral,

$$\begin{aligned} &= \frac{1}{2\pi}\int_0^{\frac{2\pi}{x}} Edt \\ &= \frac{E}{x} && (4.95) \end{aligned}$$

The adiabatic invariant is

$$I = \bar{\psi}\hat{\mathbf{x}}.\boldsymbol{\sigma}\psi. \tag{4.96}$$

5. SIMPLE SUPERSYMMETRIC LAGRANGIAN

In this chapter [7] and [18] are studied. For the Lagrangian we discussed in (4.59) an infinitesimal transformation that mixes the bosonic and fermionic degrees of freedom is given as

$$\delta_\epsilon x = \epsilon\bar{\psi} + \psi\bar{\epsilon} \tag{5.1}$$

$$\delta_\epsilon \psi = i\epsilon A(\dot{x}, x) \tag{5.2}$$

$$\delta_\epsilon \bar{\psi} = -i\bar{\epsilon}\bar{A}(\dot{x}, x) \tag{5.3}$$

where A is a complex function of $\dot{x}, x$ and ϵ is a complex Grassmann variable [14]. The variation of the Lagrangian under the transformations is

$$\delta L_\epsilon = \epsilon\bar{\psi}(-\ddot{x} + \dot{A} - V_1' - iV_2 A) + \epsilon\frac{d}{dt}(\dot{x}\bar{\psi} - \frac{1}{2}\bar{\psi}A) + \text{c.c}, \tag{5.4}$$

where "c.c." means "complex conjugation". $\bar{\psi}$ is an independent variable. The expression in the first parenthesis is zero. Consider A is a function like

$$A(\dot{x}, x) = \dot{x} + B(x) \tag{5.5}$$

and plug it in the first term on the right hand side of (5.4)

$$(B'(x) - iV_2)\dot{x} - V_1' - iV_2 B(x) = 0. \tag{5.6}$$

It is satisfied for $B(x) = iV(x)$. Therefore, the specific form of the potentials V_1 and V_2 ;

$$V_1 = \frac{1}{2}V^2(x), \quad V_2 = V'(x) \tag{5.7}$$

where $V(x)$ is an arbitrary function. Now we are left with

$$A(\dot{x}, x) = \dot{x} + iV(x). \tag{5.8}$$

The Lagrangian (4.59) becomes

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}V^2(x) + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - V'(x)\bar{\psi}\psi. \tag{5.9}$$

The equations of motion are:

$$\begin{aligned} \ddot{x} + V(x)V'(x) + V''(x)\bar{\psi}\psi &= 0 && (5.10)\\ i\dot{\psi} - V'(x)\psi &= 0. && (5.11) \end{aligned}$$

The Hamiltonian of the system is

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}V^2(x) + V'(x)\bar{\psi}\psi. \tag{5.12}$$

5.1. Supersymmetry Transformations and Conserved Charges

Under the variation of the fields Lagrangian changes by a total derivation :

$$\begin{aligned} \delta L &= \dot{x}\delta\dot{x} - V(x)V'(x)\delta x + \frac{i}{2}(\delta\bar{\psi}\dot{\psi} + \bar{\psi}\delta\dot{\psi} - \delta\dot{\bar{\psi}}\psi - \dot{\bar{\psi}}\delta\psi) - V''(x)\delta x\bar{\psi}\psi \\ &\quad -V'(x)\delta\bar{\psi}\psi - V'(x)\bar{\psi}\delta\psi && (5.13)\\ &= \epsilon(\dot{x}\frac{\dot{\bar{\psi}}}{2} + i\frac{\dot{x}}{2}V'(x)\bar{\psi} + \frac{1}{2}\ddot{x}\bar{\psi} - V(x)V'(x)\bar{\psi} + \frac{i}{2}V(x)\dot{\bar{\psi}} + V(x)V'(x)\bar{\psi}) \\ &\quad +\bar{\epsilon}(-\dot{x}\frac{\dot{\psi}}{2} + V(x)V'(x)\psi + i\frac{\dot{x}}{2}V'(x)\psi - \frac{\ddot{x}}{2}\psi + \frac{i}{2}V(x)\dot{\psi} - V(x)V'(x)\psi) \\ &= \epsilon\frac{d}{dt}(\frac{\dot{x}}{2}\bar{\psi} + \frac{i}{2}V(x)\bar{\psi}) - \bar{\epsilon}\frac{d}{dt}(\frac{\dot{x}}{2}\psi - \frac{i}{2}V(x)\psi) && (5.14) \end{aligned}$$

This is the total derivation of $F = \frac{\dot{x}}{2}\bar{\psi} + \frac{i}{2}V(x)\bar{\psi} + \text{c.c}$ in chapter 4. Since ϵ is not time dependent we can take it in the total derivation.

$$\delta S = \int \frac{d}{dt}(\epsilon(\frac{\dot{x}}{2} + \frac{i}{2}V(x))\bar{\psi})dt - \int \frac{d}{dt}(\bar{\epsilon}(\frac{\dot{x}}{2} - \frac{i}{2}V(x))\psi)dt$$

$$= \epsilon(\frac{\dot{x}}{2}\bar{\psi} + \frac{i}{2}V(x)\bar{\psi}) \mid_{-\infty}^{+\infty} - \bar{\epsilon}(\frac{\dot{x}}{2}\psi - \frac{i}{2}V(x)\psi) \mid_{-\infty}^{+\infty} = 0 \tag{5.15}$$

Due to the zero change of action under the variation of the fields, the system has a symmetry related to the transformations above which the variation parameter is fermion. Such a transformation is a fermionic symmetry where δ_{ϵ_i} is the fermionic transformation. Now, we take the variation parameter depend on time $\epsilon = \epsilon(t)$. Since ϵ and $\bar{\epsilon}$ are two independent parameters, there are two associated integrals of motion.

$$\begin{aligned} \delta\dot{x} &= \dot{\epsilon}\overline{\psi} + \epsilon\dot{\overline{\psi}} - \dot{\bar{\epsilon}}\psi - \bar{\epsilon}\dot{\psi} && (5.16)\\ \delta\dot{\psi} &= \dot{\epsilon}(i\dot{x} + V(x)) + \epsilon(i\ddot{x} + V'(x)\dot{x}) && (5.17)\\ \delta\dot{\overline{\psi}} &= \dot{\bar{\epsilon}}(-i\dot{x} + V(x)) + \bar{\epsilon}(-i\ddot{x} + V'(x)\dot{x}) && (5.18) \end{aligned}$$

Under the variation of the time dependent fields Lagrangian changes as

$$\begin{aligned} \delta L &= \dot{x}(\dot{\epsilon}\overline{\psi} + \epsilon\dot{\overline{\psi}} - \dot{\bar{\epsilon}}\psi - \bar{\epsilon}\dot{\psi}) - V(x)V'(x)(\epsilon\overline{\psi} - \bar{\epsilon}\psi) - V''(x)(\epsilon\overline{\psi} - \bar{\epsilon}\psi)\overline{\psi}\psi \\ &+\frac{i}{2}(\bar{\epsilon}(-i\dot{x} + V(x))\dot{\psi} + \overline{\psi}(i\dot{\epsilon}\dot{x} + i\epsilon\ddot{x} + \dot{\epsilon}V(x) + \epsilon V'(x)\dot{x}) \\ &-\frac{i}{2}((-i\dot{\bar{\epsilon}}\dot{x} - i\bar{\epsilon}\ddot{x} + \dot{\bar{\epsilon}}V(x) + \bar{\epsilon}V'(x)\dot{x})\psi + \dot{\overline{\psi}}\epsilon(i\dot{x} + V(x))) \\ &-V'(x)\bar{\epsilon}(-i\dot{x} + V(x))\psi - V'(x)\overline{\psi}\epsilon(i\dot{x} + V(x)). \end{aligned} \tag{5.19}$$

Let's separate it to some parts to collect similar terms. Firstly, $\epsilon\overline{\psi}$ and $\bar{\epsilon}\psi$ terms cancel:

$$\Rightarrow \quad \epsilon\overline{\psi}(-V(x)V'(x) + V(x)V'(x)) + \bar{\epsilon}\psi(V(x)V'(x) - V(x)V'(x)) = 0$$

Now we get some total derivations :

$$\begin{aligned}
&\Rightarrow && -\frac{i}{2}\dot{\overline{\psi}}\epsilon V(x) - i\overline{\psi}\epsilon V'(x)\dot{x} + \frac{i}{2}\overline{\psi}\epsilon V'(x)\dot{x} + \frac{i}{2}\overline{\psi}\dot{\epsilon}V(x) \\
&= && -\frac{i}{2}\frac{d}{dt}(\overline{\psi}V(x)\epsilon) + i\overline{\psi}\dot{\epsilon}V(x) \\
&\Rightarrow && \dot{x}\dot{\epsilon}\overline{\psi} - \frac{1}{2}\overline{\psi}\dot{\epsilon}\dot{x} - \frac{1}{2}\overline{\psi}\epsilon\ddot{x} + \frac{1}{2}\dot{\overline{\psi}}\epsilon\dot{x} + \dot{x}\epsilon\dot{\overline{\psi}} \\
&= && -\frac{3}{2}\overline{\psi}\dot{\epsilon}\dot{x} - \frac{1}{2}\frac{d}{dt}(\overline{\psi}\dot{x}).\epsilon = -\frac{1}{2}\frac{d}{dt}(\overline{\psi}\dot{x}\epsilon) - \overline{\psi}\dot{\epsilon}\dot{x} \\
&\Rightarrow && -\frac{1}{2}\dot{x}\overline{\epsilon}\dot{\psi} - \frac{1}{2}\overline{\epsilon}\ddot{x}\psi - \dot{x}\dot{\overline{\epsilon}}\psi - \frac{1}{2}\dot{\overline{\epsilon}}\dot{x}\psi + \frac{i}{2}\overline{\epsilon}V(x)\dot{\psi} - \frac{i}{2}\overline{\epsilon}V'(x)\dot{x}\psi + i\overline{\epsilon}\psi V'(x)\dot{x} \\
& && -\frac{i}{2}\dot{\overline{\epsilon}}V(x)\psi \\
&= && -\frac{1}{2}\frac{d}{dt}(\overline{\epsilon}\dot{x}\psi) - \dot{x}\dot{\overline{\epsilon}}\psi + \frac{i}{2}\overline{\epsilon}\frac{d}{dt}(V(x)\psi) + \frac{i}{2}\dot{\overline{\epsilon}}V(x)\psi - i\dot{\overline{\epsilon}}V(x)\psi
\end{aligned}$$

After plugging them in the Lagrangian, one finds

$$\begin{aligned}
\delta L &= -\frac{i}{2}\frac{d}{dt}(\overline{\psi}V(x)\epsilon) - \frac{1}{2}\frac{d}{dt}(\overline{\psi}\dot{x}\epsilon) - \frac{1}{2}\frac{d}{dt}(\overline{\epsilon}\dot{x}\psi) + \frac{i}{2}\frac{d}{dt}(\overline{\epsilon}V(x)\psi) + i\overline{\psi}\dot{\epsilon}V(x) \\
&\quad -\overline{\psi}\dot{\epsilon}\dot{x} - \dot{x}\dot{\overline{\epsilon}}\psi - i\dot{\overline{\epsilon}}V(x)\psi \qquad (5.20) \\
\delta S &= \{-\frac{i}{2}\overline{\psi}V(x)\epsilon - \frac{1}{2}\overline{\psi}\dot{x}\epsilon - \frac{1}{2}\overline{\epsilon}\dot{x}\psi + \frac{i}{2}\overline{\epsilon}V(x)\psi\}|_{-\infty}^{+\infty} \\
&\quad + \int\{-i\dot{\epsilon}\overline{\psi}(i\dot{x} + V(x)) - i\dot{\overline{\epsilon}}\psi(-i\dot{x} + V(x))\}dt \qquad (5.21)
\end{aligned}$$

First term is zero. Now we are left with

$$= \int(\dot{\epsilon}Q - \dot{\overline{\epsilon}}\overline{Q})dt. \qquad (5.22)$$

where

$$Q = \bar{\psi}(\dot{x} - iV(x)), \qquad \bar{Q} = (\dot{x} + iV(x))\psi \qquad (5.23)$$

are super charges associated with the supersymmetry. In addition to this way of finding charges, they can also be obtained from the formula (4.40). Since ϵ and $\bar{\epsilon}$ are Grassmann variables, super charges are Grassmann odd functions. (5.23) can be rewritten in terms of canonical coordinates (x,p) and (ψ,π_ψ) as

$$Q=\pi_\psi(p-iV(x)), \qquad \bar{Q}=\psi(-ip+V(x)). \tag{5.24}$$

They form an algebra with respect to the Poisson bracket. We can find the Hamiltonian by using the commutator of the super charges

$$\begin{aligned}\{Q,\bar{Q}\} &= \frac{\partial Q}{\partial x}\frac{\partial \bar{Q}}{\partial p}+\frac{\partial Q}{\partial \psi}\frac{\partial \bar{Q}}{\partial \pi_\psi}-\frac{\partial Q}{\partial p}\frac{\partial \bar{Q}}{\partial x}+\frac{\partial Q}{\partial \pi_\psi}\frac{\partial \bar{Q}}{\partial \psi} \qquad (5.25)\\ &= -i\pi_\psi V'(x)(-i\psi)-\pi_\psi\psi V'(x)-i(ip+V(x))(-ip+V(x))\\ &= -i(2V'(x)\bar{\psi}\psi+p^2+V(x)^2)\\ &= -2iH.\end{aligned}$$

In other words, Hamiltonian is related to Noether charges with the formula

$$H=\frac{i}{2}\{Q,\bar{Q}\}=\frac{i}{2}\{\bar{Q},Q\}. \tag{5.26}$$

and it is automatic that

$$\{Q,H\}=\{\bar{Q},H\}=0. \tag{5.27}$$

In addition to the Poisson bracket method, another way to get formula as an algebra is the infinitesimal transformation of the supersymmetric generator. For any arbitrary function fermionic transformation is given as

$$\delta_\epsilon f=i\{\epsilon Q+\bar{\epsilon}\bar{Q},f\}. \tag{5.28}$$

Here are some examples that hold with our transformations;

$$\begin{aligned}\delta_\epsilon x &= i(\epsilon\{Q,x\}+\bar{\epsilon}\{\bar{Q},x\}) \qquad (5.29)\\ &= i(\epsilon(-\frac{\partial Q}{\partial p})+\bar{\epsilon}(-\frac{\partial \bar{Q}}{\partial p}))=-i\epsilon\pi_\psi+i\bar{\epsilon}(i\psi)\\ &= \epsilon\bar{\psi}-\bar{\epsilon}\psi \qquad (5.30)\end{aligned}$$

$$\begin{aligned}
\delta_\epsilon \psi &= i\epsilon\{Q,\psi\} + i\bar{\epsilon}\{\bar{Q},\psi\} && (5.31)\\
&= i\epsilon\frac{\partial Q}{\partial \pi_\psi} + i\bar{\epsilon}\frac{\partial \bar{Q}}{\partial \pi_\psi} = i\epsilon(p - iV(x))\\
&= \epsilon(i\dot{x} + V(x)) && (5.32)\\
\delta_\epsilon \bar{\psi} &= \bar{\epsilon}(-i\dot{x} + V(x)) && (5.33)
\end{aligned}$$

Time translation in the Lagrangian corresponds to the Hamiltonian as a Noether charge:

$$\begin{aligned}
\delta x &= \dot{x}s, \quad \delta\dot{x} = \ddot{x}s && (5.34)\\
\delta \psi &= \dot{\psi}s, \quad \delta\dot{\psi} = \ddot{\psi}s && (5.35)\\
\delta \bar{\psi} &= \dot{\bar{\psi}}s, \quad \delta\dot{\bar{\psi}} = \ddot{\bar{\psi}}s && (5.36)
\end{aligned}$$

Note that s doesn't depend on time. Variation of the Lagrangian under the given transformations :

$$\begin{aligned}
\delta L &= \dot{x}\delta\dot{x} - V(x)V'(x)\delta x + \frac{i}{2}(\delta\bar{\psi}\dot{\psi} + \bar{\psi}\delta\dot{\psi} - \delta\dot{\bar{\psi}}\psi - \dot{\bar{\psi}}\delta\psi) - V''(x)\delta x\bar{\psi}\psi\\
&\quad -V'(x)\delta\bar{\psi}\psi - V'(x)\bar{\psi}\delta\psi\\
&= s(\frac{d}{dt}(\frac{\dot{x}^2}{2} - \frac{V(x)^2}{2}) + \frac{i}{2}\frac{d}{dt}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \frac{d}{dt}(V'(x)\bar{\psi}\psi)) && (5.37)\\
\delta S &= \int \frac{d}{dt}s(\frac{\dot{x}^2}{2} - \frac{V(x)^2}{2} + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - V'(x)\bar{\psi}\psi)dt\\
&= s(\frac{\dot{x}^2}{2} - \frac{V(x)^2}{2} + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - V'(x)\bar{\psi}\psi)|_{-\infty}^{+\infty} = 0 && (5.38)
\end{aligned}$$

Assume s is time dependent $s = s(t)$, then the variation of the fields are

$$\begin{aligned}
\delta\dot{x} &= \ddot{x}s + \dot{x}\dot{s} && (5.39)\\
\delta\dot{\psi} &= \ddot{\psi}s + \dot{\psi}\dot{s} && (5.40)\\
\delta\dot{\bar{\psi}} &= \ddot{\bar{\psi}}s + \dot{\bar{\psi}}\dot{s} && (5.41)
\end{aligned}$$

Under this variation of fields the Lagrangian changes as $\delta L = s(t)\frac{d}{dt}(f) + \dot{s}(t)(h)$.

$$\delta S = \int \dot{s}(h - f)dt. \tag{5.42}$$

where f stands for the terms differentiated with respect to time and h comes with $\dot{s}$ terms. We already know f from the variation of the Lagrangian (5.37). It is enough to collect the terms with $\dot{s}$ in the δL to find h .

$$\delta L = \dot{x}\delta\dot{x} + \frac{i}{2}(\bar{\psi}\delta\dot{\psi} - \delta\dot{\bar{\psi}}\psi)$$

$$= \dot{s}(\dot{x}^2 + \frac{i}{2}\bar{\psi}\dot{\psi} - \frac{i}{2}\dot{\bar{\psi}}\psi) \tag{5.43}$$

$$\delta S = \int \dot{s}(\dot{x}^2 + \frac{i}{2}\bar{\psi}\dot{\psi} - \frac{i}{2}\dot{\bar{\psi}}\psi - \frac{\dot{x}^2}{2} + \frac{V(x)^2}{2} - \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) + V'(x)\bar{\psi}\psi)dt \tag{5.44}$$

The Noether charge is

$$H = \frac{\dot{x}^2}{2} + \frac{V(x)^2}{2} + V'(x)\bar{\psi}\psi \tag{5.45}$$

This is the Hamiltonian of the system associated to time translation as a conserved charge. Similar to the case of fermionic transformation, time transformation is related to the Hamiltonian in the following way:

$$\begin{aligned}
\delta_s f &= -\{sH, f\} && (5.46)\\
&= -s(\frac{\partial H}{\partial x}\frac{\partial f}{\partial p} + \frac{\partial H}{\partial \psi}\frac{\partial f}{\partial \pi_\psi} - \frac{\partial H}{\partial p}\frac{\partial f}{\partial x} + \frac{\partial H}{\partial \pi_\psi}\frac{\partial f}{\partial \psi}) \\
&= -s(V(x)V'(x)\frac{\partial f}{\partial p} - iV'(x)\pi_\psi\frac{\partial f}{\partial \pi_\psi} - p\frac{\partial f}{\partial x} - iV'(x)\psi\frac{\partial f}{\partial \psi}) && (5.47)
\end{aligned}$$

When we take the variation of x and ψ under time translation,

$$\begin{aligned}
\delta_s x &= s\frac{\partial H}{\partial p} = s\dot{x} \\
\delta_s \psi &= -s\frac{\partial H}{\partial \pi_\psi} = -isV'(x)\psi = s\dot{\psi}
\end{aligned}$$

they satisfy both equations of motion (4.61) and transformations (5.34)-(5.36).

Now, we compute the commutators of supersymmetric transformations. The relation between the fermionic transformation and time translation is

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]f = \delta_s f. \tag{5.48}$$

We mean that once fermionic transformation is applied twice, one gets a time translation. They act on a bosonic variable, i.e. x

$$\begin{aligned}
[\delta_1, \delta_2]x &= \delta_1(\delta_2 x) - \delta_2(\delta_1 x) \qquad (5.49) \\
&= \epsilon_2\delta_1\bar{\psi} - \bar{\epsilon_2}\delta_1\psi - \epsilon_1\delta_2\bar{\psi} + \bar{\epsilon_1}\delta_2\psi \\
&= i\dot{x}(\bar{\epsilon_1}\epsilon_2 + \epsilon_1\bar{\epsilon_2} - \epsilon_2\bar{\epsilon_1} - \bar{\epsilon_2}\epsilon_1) + V(x)(\epsilon_2\bar{\epsilon_1} - \bar{\epsilon_2}\epsilon_1 - \epsilon_1\bar{\epsilon_2} + \bar{\epsilon_1}\epsilon_2)
\end{aligned}$$

We use the anti-commutation relation of Grassmann variables $\epsilon_1\epsilon_2 + \epsilon_2\epsilon_1 = 0$ so one finds

$$[\delta_1, \delta_2]x = 2i(\epsilon_1\bar{\epsilon_2} - \epsilon_2\bar{\epsilon_1})\dot{x} \tag{5.50}$$

Now commutator of two supersymmetric transformations acting on a fermion is obtained in the following way:

$$
\begin{aligned}
[\delta_1, \delta_2]\psi &= \delta_1(\delta_2\psi) - \delta_2(\delta_1\psi) \\
&= i\epsilon_2(\epsilon_1\dot{\bar{\psi}} - \bar{\epsilon_1}\dot{\psi}) + \epsilon_2 V'(x)(\epsilon_1\bar{\psi} - \bar{\epsilon_1}\psi) - i\epsilon_1(\epsilon_2\dot{\bar{\psi}} - \bar{\epsilon_2}\dot{\psi}) \\
&\quad -\epsilon_1 V'(x)(\epsilon_2\bar{\psi} - \bar{\epsilon_2}\psi) \\
&= \epsilon_2\epsilon_1(i\dot{\bar{\psi}} + V'(x)\bar{\psi}) - \epsilon_1\epsilon_2(i\dot{\bar{\psi}} + V'(x)\bar{\psi}) - \epsilon_2\bar{\epsilon_1}(i\dot{\psi} + V'(x)\psi) \\
&\quad +\epsilon_1\bar{\epsilon_2}(i\dot{\psi} + V'(x)\psi)
\end{aligned} \tag{5.51}
$$

By using the equations of motion in (4.61), some terms cancel each other and we get

$$[\delta_1, \delta_2]\psi = 2i(\epsilon_1\bar{\epsilon_2} - \epsilon_2\bar{\epsilon_1})\dot{\psi}. \tag{5.52}$$

As one can see, here the commutator of two supersymmetric transformations equals to the time translation. We can see the relation between them is $s = 2i(\epsilon_1\bar{\epsilon_2} - \epsilon_2\bar{\epsilon_1})$. (See Appendix C.)

6. D-PARTICLE IN A BACKGROUND MAGNETIC FIELD

In [19] a Lagrangian which describes the classical system of D-particles is introduced

$$L = \frac{m}{2}(\dot{\mathbf{x}}^2 + D^2 + 2i\bar{\lambda}\dot{\lambda}) - U(\mathbf{x})D + \mathbf{A}(\mathbf{x}).\dot{\mathbf{x}} + C(\mathbf{x})\bar{\lambda}\lambda + \mathbf{C}(\mathbf{x}).\bar{\lambda}\boldsymbol{\sigma}\lambda \tag{6.1}$$

where the position coordinate $\mathbf{x} = (x(t), y(t), z(t))$ is bosonic field together with its fermionic super partner, given by a 2- component spinor λ_α, $\alpha = 1,2$ and its complex conjugate $\bar{\lambda}^\alpha = (\lambda_\alpha)^*$. D is an auxiliary bosonic variable. $\mathbf{A}(\mathbf{x}), C(\mathbf{x}), \mathbf{C}(\mathbf{x})$ and $U(\mathbf{x})$ are background fields. $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli spin matrices. In this section we review the model, its physical interpretation and some properties related to supersymmetry.

6.1. The Lagrangian

The term $\mathbf{A}(\mathbf{x}).\dot{\mathbf{x}}$ in L is a magnetic coupling term which yields a Lorentz-type force that is $\mathbf{F}_{\text{Lorentz}} \sim \dot{\mathbf{x}} \times \mathbf{B}$ and the first term is a position-dependent term. Now we will find equations of motion and the Hamiltonian of the system. There is no $\dot{D}$ term so we easily find

$$D = \frac{U(x)}{m}. \tag{6.2}$$

Plugging D in the Lagrangian, we have

$$L = \frac{m}{2}(\dot{\mathbf{x}}^2 + 2i\bar{\lambda}\dot{\lambda}) - \frac{U^2(\mathbf{x})}{2m} + \mathbf{A}(\mathbf{x}).\dot{\mathbf{x}} + C(\mathbf{x})\bar{\lambda}\lambda + \mathbf{C}(\mathbf{x}).\bar{\lambda}\boldsymbol{\sigma}\lambda \tag{6.3}$$

Equations of motion are :

$$m\ddot{x}_j + (\partial_j A_i - \partial_i A_j)\dot{x}_j + \partial_j \frac{U^2(\mathbf{x})}{2m} - \partial_j C(x)\bar{\lambda}\lambda - \partial_j C_i(x)\bar{\lambda}\boldsymbol{\sigma}_i\lambda = 0 \qquad (6.4)$$

$$\dot{\lambda} - \frac{i}{m}C(\mathbf{x})\lambda - \frac{i}{m}C_i(\mathbf{x})\sigma_i\lambda = 0. \qquad (6.5)$$

The canonical momenta for both bosonic field and spinors are

$$p = m\dot{x}_i + A_i(x), \qquad \pi_\lambda = im\bar{\lambda}. \qquad (6.6)$$

The Hamiltonian of the system

$$H = \frac{m}{2}\dot{x}_i^2 + \frac{U^2(\mathbf{x})}{2m} - C(\mathbf{x})\bar{\lambda}\lambda - C_i(x)\bar{\lambda}\sigma_i\lambda. \qquad (6.7)$$

6.2. Supersymmetry Conservation

We use some ordering for various terms in the Lagrangian to check the supersymmetry :

$$O(\mathbf{x}) = 0, \quad O(\frac{d}{dt}) = 1, \quad O(\xi) = -1/2, \quad O(D) = 1, \quad O(\lambda) = 1/2$$

so that the order is conserved in the supersymmetry transformations. Let us split the Lagrangian

$$L^{(1)} = \frac{1}{2}m(\dot{\mathbf{x}}^2 + D^2 + 2i\bar{\lambda}\dot{\lambda}) \qquad (6.8)$$

$$L^{(2)} = -U(\mathbf{x})D + \mathbf{A}(\mathbf{x}).\dot{\mathbf{x}} + C(\mathbf{x})\bar{\lambda}\lambda + \mathbf{C}(\mathbf{x}).\bar{\lambda}\boldsymbol{\sigma}\lambda \qquad (6.9)$$

The corresponding transformations are given in the following form:

$$\begin{aligned}
\delta \mathbf{x} &= i\bar{\lambda}\boldsymbol{\sigma}\xi - i\bar{\xi}\boldsymbol{\sigma}\lambda && (6.10)\\
\delta \lambda &= \dot{\mathbf{x}}.\boldsymbol{\sigma}\xi + iD\xi && (6.11)\\
\delta D &= -\dot{\bar{\lambda}}\xi - \bar{\xi}\dot{\lambda} && (6.12)
\end{aligned}$$

Each part of the Lagrangian $L^{(i)}$ (6.8) and (6.9) has to be supersymmetric to obtain a totally supersymmetric Lagrangian. Firstly, we search for the $L^{(1)}$ invariance under the transformations.

$$\delta L^{(1)} = m(\dot{x}_i\delta\dot{x}_i + D\delta D + i\delta\bar{\lambda}\dot{\lambda} + i\bar{\lambda}\delta\dot{\lambda}) \tag{6.13}$$

The variation of the Lagrangian under the transformations

$$\delta L^{(1)} = im\dot{x}_i\dot{\bar{\lambda}}\sigma_i\xi - im\dot{x}_i\bar{\xi}\sigma_i\dot{\lambda} - mD\dot{\bar{\lambda}}\xi - mD\bar{\xi}\dot{\lambda} + im(\bar{\xi}\sigma_i\dot{x}_i - i\bar{\xi}D)\dot{\lambda} + im\bar{\lambda}(\ddot{x}_i\sigma_i\xi + i\dot{D}\xi) \tag{6.14}$$

yields some total derivatives

$$\delta L^{(1)} = im\frac{d}{dt}(\dot{x}_i\bar{\lambda})\sigma_i\xi - m\frac{d}{dt}(D\bar{\lambda})\xi. \tag{6.15}$$

When plugged in the variation of the action, it gives zero.

$$\delta S^{(1)} = \int \delta L^{(1)}dt = im\dot{x}_i\bar{\lambda}\sigma_i\xi|_{-\infty}^{+\infty} - mD\bar{\lambda}\xi|_{-\infty}^{+\infty} = 0 \tag{6.16}$$

Now, we apply the supersymmetry transformations on the $L^{(2)}$ Lagrangian. It requires some constraints on U, $\mathbf{A}$, C and $\mathbf{C}$ to make $L^{(2)}$ supersymmetric.

$$\begin{aligned}
\delta L^{(2)} &= -\partial_i U(\mathbf{x})\delta x_i D - U(x)\delta D + \partial_j A_i(x)\delta x_j\dot{x}_i + A_i(x)\delta\dot{x}_i + \partial_i C(\mathbf{x})\delta x_i\bar{\lambda}\lambda\\
&\quad + C(\mathbf{x})\delta\bar{\lambda}\lambda + C(\mathbf{x})\bar{\lambda}\delta\lambda + \partial_j C_i(\mathbf{x})\delta x_j\bar{\lambda}\sigma_i\lambda + C_i(\mathbf{x})\delta\bar{\lambda}\sigma_i\lambda\\
&\quad + C_i(\mathbf{x})\bar{\lambda}\sigma_i\delta\lambda. && (6.17)
\end{aligned}$$

We insert the supersymmetric transformations:

$$\begin{aligned}
= \;& -\partial_i U(x)(i\bar{\lambda}\sigma_i\xi - i\bar{\xi}\sigma_i\lambda)D - U(x)(-\dot{\bar{\lambda}}\xi - \bar{\xi}\dot{\lambda}) + \partial_j A_i(x)(i\bar{\lambda}\sigma_j\xi - i\bar{\xi}\sigma_j\lambda)\dot{x}_i \\
& + A_i(x)(i\dot{\bar{\lambda}}\sigma_i\xi - i\bar{\xi}\sigma_i\dot{\lambda}) + \partial_i C(x)(i\bar{\lambda}\sigma_i\xi - i\bar{\xi}\sigma_i\lambda)\bar{\lambda}\lambda + C(x)(\bar{\xi}\sigma_i\dot{x}_i - i\bar{\xi}D)\lambda \\
& + C(x)\bar{\lambda}(\dot{x}_i\sigma_i\xi + iD\xi) + \partial_j C_i(x)(i\bar{\lambda}\sigma_j\xi - i\bar{\xi}\sigma_j\lambda)\bar{\lambda}\sigma_i\lambda + C_i(x)(\bar{\xi}\sigma_j\dot{x}_j - i\bar{\xi}D)\sigma_i\lambda \\
& + C_i(x)\bar{\lambda}\sigma_i(\dot{x}_j\sigma_j\xi + iD\xi) \qquad (6.18)
\end{aligned}$$

Firstly, we collect the terms with D and equal them to zero.

$$iD\bar{\xi}\lambda(\partial_i U(\mathbf{x})\sigma_i - C(\mathbf{x}) - C_i(\mathbf{x})\sigma_i) + c.c = 0 \qquad (6.19)$$

The expression in the parenthesis must be zero

$$\partial_i U(\mathbf{x})\sigma_i - C(\mathbf{x}) - C_i(\mathbf{x})\sigma_i = 0 \qquad (6.20)$$

It is satisfied when $C(\mathbf{x}) = 0$ and $C_i(\mathbf{x}) = \partial U/\partial x_i$ or in general

$$\mathbf{C} = \boldsymbol{\nabla} U. \qquad (6.21)$$

After dropping $C(\mathbf{x})$ term , the variation becomes

$$\begin{aligned}
\delta L^{(2)} = \;& \dot{\bar{\lambda}}\left(U(\mathbf{x}) + iA_i\sigma_i\right)\xi + \bar{\xi}(U(\mathbf{x}) - iA_i\sigma_i)\dot{\lambda} + i\bar{\lambda}\sigma_j\xi(\partial_j A_i\dot{x}_i + \partial_j C_i\bar{\lambda}\sigma_i\lambda) \\
& - i\bar{\xi}\sigma_j\lambda(\partial_j A_i\dot{x}_i + \partial_j C_i\bar{\lambda}\sigma_i\lambda) + C_i(x)\dot{x}_j(\bar{\xi}\sigma_j\sigma_i\lambda + \bar{\lambda}\sigma_i\sigma_j\xi). \qquad (6.22)
\end{aligned}$$

We rewrite the Lagrangian in terms of total derivations

$$\begin{aligned}
= \;& \frac{d}{dt}\left(\bar{\lambda}(U(\mathbf{x}) + iA_i\sigma_i)\xi + \bar{\xi}(U(\mathbf{x}) - iA_i\sigma_i)\lambda\right) - \bar{\lambda}(\partial_l U(\mathbf{x})\dot{x}_l + i\partial_l A_i\dot{x}_l\sigma_i)\xi \\
& - \bar{\xi}(\partial_l U(\mathbf{x})\dot{x}_l - i\partial_l A_i\dot{x}_l\sigma_i)\lambda + (i\bar{\lambda}\sigma_j\xi - i\bar{\xi}\sigma_j\lambda)(\partial_j A_l\dot{x}_l + \partial_j C_i(\mathbf{x})\bar{\lambda}\sigma_i\lambda) \\
& + C_i(\mathbf{x})\dot{x}_l(\bar{\xi}\sigma_l\sigma_i\lambda + \bar{\lambda}\sigma_i\sigma_l\xi). \qquad (6.23)
\end{aligned}$$

After collecting terms with $\dot{x}_l$ and using the Pauli matrices identity

$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \mathbf{1}\delta_{ij}, \tag{6.24}$$

variation of the Lagrangian takes the form:

$$\begin{aligned} = \quad & \dot{x}_l((-\bar{\lambda}\partial_l U(\mathbf{x})\dot{x}_l - i\bar{\lambda}\partial_l A_j\sigma_j)\xi - \bar{\xi}(\partial_l U(\mathbf{x})\dot{x}_l - i\partial_l A_j\sigma_j)\lambda + i\bar{\lambda}\sigma_j\partial_j A_l\xi \\ & -i\bar{\xi}\sigma_j\partial_j A_l\lambda + \bar{\xi}(-i\epsilon_{ilj}C_i(\mathbf{x})\sigma_j + C_l(\mathbf{x}))\lambda + \bar{\lambda}(i\epsilon_{ilj}C_i(\mathbf{x})\sigma_j + C_l(\mathbf{x}))\xi) \\ & +i\bar{\lambda}\sigma_j\partial_j C_i(\mathbf{x})(\bar{\lambda}\sigma_i\lambda)\xi - i\bar{\xi}\sigma_j\partial_j C_i(\mathbf{x})(\bar{\lambda}\sigma_i\lambda)\lambda \end{aligned} \tag{6.25}$$

We collect the terms with $\bar{\lambda}\xi, \bar{\xi}\lambda, \bar{\lambda}\sigma_j\xi, \bar{\xi}\sigma_j\lambda$ in $\dot{x}_l$ term and equal them to zero.

$$(\bar{\lambda}\xi + \bar{\xi}\lambda)(-\partial_l U(\mathbf{x}) + C_l(x)) + i\bar{\lambda}(-\partial_l A_j + \partial_j A_l + \varepsilon_{ilj}C_i(\mathbf{x}))\sigma_j\xi$$

$$-i\bar{\xi}(-\partial_l A_j + \partial_j A_l + \varepsilon_{ilj}C_i(\mathbf{x}))\sigma_j\lambda = 0 \tag{6.26}$$

The first term vanishes due to (6.21). The expressions in the parenthesis should also be zero. Therefore, we get

$$\varepsilon_{ilj}C_i(\mathbf{x}) = \partial_l A_j - \partial_j A_l \tag{6.27}$$

We use the identity $\varepsilon_{ilj}\varepsilon_{klj} = 2\delta_{ik}$ and obtain

$$C_k(\mathbf{x}) = \frac{1}{2}\varepsilon_{klj}(\partial_l A_j - \partial_j A_l) \tag{6.28}$$

where $F_{lj} = \partial_l A_j(x) - \partial_j A_l(x)$. It is related to the magnetic field $\mathbf{B}$ with the formula

$$B_k = \frac{1}{2}\varepsilon_{klj}F_{lj}. \tag{6.29}$$

More generally, we can say

$$\mathbf{B} = \mathbf{C} = \nabla \times \mathbf{A}. \tag{6.30}$$

For that reason, the second and third terms vanish in (6.26). Assume that

$$z = \bar{\lambda}\sigma_j \xi \bar{\lambda}\sigma_i \lambda. \tag{6.31}$$

Then the remaining terms in (6.25) are written again by using $z - z^\dagger = 2i\mathrm{Im}(z)$:

$$i\partial_j C_i(z - z^\dagger) = -2\partial_j C_i \mathrm{Im}(\bar{\lambda}\sigma_j \xi \bar{\lambda}\sigma_i \lambda) \tag{6.32}$$

Firstly, z is written explicitly

$$\bar{\lambda}^\alpha (\sigma_j)^\beta_\alpha \xi_\beta \bar{\lambda}^\gamma (\sigma_i)^\delta_\gamma \lambda_\delta = \bar{\lambda}^\alpha \xi_\beta \bar{\lambda}^\gamma \lambda_\delta (\sigma_j)^\beta_\alpha (\sigma_i)^\delta_\gamma. \tag{6.33}$$

It is important to change the sign when the order of fermions changes

$$= -\bar{\lambda}^\alpha \bar{\lambda}^\gamma \xi_\beta \lambda_\delta (\sigma_j)^\beta_\alpha (\sigma_i)^\delta_\gamma. \tag{6.34}$$

Use the definition

$$\bar{\lambda}^\alpha \bar{\lambda}^\gamma = -\bar{\lambda}^\gamma \bar{\lambda}^\alpha = \frac{1}{2}\epsilon_{\mu\nu}\bar{\lambda}^\mu \bar{\lambda}^\nu \epsilon^{\alpha\gamma} = |\bar{\lambda}|^2 \epsilon^{\alpha\gamma} \tag{6.35}$$

so we find

$$z = -|\bar{\lambda}|^2 \epsilon^{\alpha\gamma} \xi_\beta (\sigma_j)^\beta_\alpha (\sigma_i)^\delta_\gamma \lambda_\delta. \tag{6.36}$$

Let's define a matrix called $M_{ij} = \epsilon^{\alpha\gamma}(\sigma_j)^{\beta}_{a}(\sigma_i)^{\delta}_{\gamma}$ and also remember that $C_i = \partial_i U$. Then,(6.32) becomes

$$2|\bar{\lambda}|^2\lambda_\delta\xi_\beta\partial_j\partial_i U \mathrm{Im}(M_{ij}) \tag{6.37}$$

We can split the matrix M_{ij} into the symmetric and antisymmetric parts:

$$M_{ij} = M_{[ij]} + M_{(ij)} \tag{6.38}$$

where $M_{[ij]}$ is antisymmetric and $M_{(ij)}$ is symmetric part of the matrix. Using the same way, one realizes that $\partial_j C_i = \partial_{(j}C_{i)}$ is a symmetric matrix since $\partial_{[j}C_{i]} = \partial_{[j}\partial_{i]}U = 0$.

$$\partial_{(j}\partial_{i)}UM_{ij} = \partial_{(j}\partial_{i)}U(M_{[ij]} + M_{(ij)}) \tag{6.39}$$

The first term on the right hand side is zero. We can write $M_{(ij)} = M_{\{ij\}} + \frac{1}{3}M_{kk}\delta_{ij}$. Using the previous results $\partial_j C_i = \partial_j\epsilon_{ikl}\partial_k A_l$, we get

$$\partial_{(j}\partial_{i)}UM_{(ij)} \quad = \quad \partial_j\epsilon_{ikl}\partial_k A_l M_{\{ij\}} + \frac{1}{3}\partial_j\epsilon_{ikl}\partial_k A_l M_{ii}\delta_{ij} \tag{6.40}$$

The second term is zero because $\partial_i\epsilon_{ikl}\partial_k A_l = 0$. Now we want to find $\partial_j C_i M_{\{ij\}}$. The trace of $M_{\{ij\}}$ is zero since $M_{\{ii\}} = M_{(ii)} - \frac{1}{3}M_{kk}.3 = 0$. Define $\epsilon^{\alpha\gamma}(\sigma_i)_\gamma = \epsilon\sigma_i$ and rewrite $M_{ij} = \sigma_j^T\epsilon\sigma_i$.

$$\sigma^T_{\{j}\epsilon\sigma_{i\}} = \sigma^T_{(j}\epsilon\sigma_{i)} - \frac{1}{3}\sigma^T_{(i}\epsilon\sigma_{i)}\delta_{ij}. \tag{6.41}$$

We use the property $\sigma^T_{(j}\epsilon\sigma_{i)} = -\epsilon\delta_{ij}$:

$$= -\epsilon\delta_{ij} - \frac{1}{3}(-3\epsilon)\delta_{ij} = 0 \tag{6.42}$$

and the result is $M_{\{ij\}} = 0$. Putting this in the equation finally we have

$$\partial_j C_i M_{\{ij\}} = 0. \tag{6.43}$$

In conclusion, supersymmetry invariance gives us

$$\mathbf{C} = \boldsymbol{\nabla} U = \boldsymbol{\nabla} \times \mathbf{A}, \qquad C = 0. \tag{6.44}$$

When we take the divergence of $\mathbf{C}$, we immediately see it is zero up to singular points.

$$\boldsymbol{\nabla}.\mathbf{C} = \boldsymbol{\nabla}^2 U = \boldsymbol{\nabla}.(\boldsymbol{\nabla} \times \mathbf{A}) = 0$$

We can find a general harmonic solution for U which is

$$U = \sum_i \frac{a_i}{r_i} + b. \tag{6.45}$$

6.2.1. A Specific Example

A special form of the Lagrangian (6.1) is

$$L = \frac{m}{2}(\dot{\mathbf{x}}^2 + D^2 + 2i\bar{\lambda}\dot{\lambda}) - (\frac{\kappa}{2|\mathbf{x}|} + \theta)D - \kappa \mathbf{A}^d.\dot{\mathbf{x}} - \frac{\kappa\, \mathbf{x}}{2|\mathbf{x}|^3}.\bar{\lambda}\boldsymbol{\sigma}\lambda \tag{6.46}$$

where the spherical symmetric solution is with $r = |\mathbf{x}|$ and the special form of (6.45) is

$$U = \frac{\kappa}{2r} + \theta, \qquad\qquad \mathbf{A} = -\kappa \mathbf{A}^d. \tag{6.47}$$

θ and κ are constants and $\mathbf{A}^d$ is a Dirac monopole vector potential in section B.3. The explicit form of the magnetic coupling term is computed for the north magnetic monopole vector potential:

$$\begin{aligned}\mathbf{A}^N.\dot{\mathbf{x}} &= -\frac{gy}{r(r+z)}\dot{x} + \frac{gx}{r(r+z)}\dot{y} \\ &= g(\frac{-\dot{x}y + x\dot{y}}{r^2+rz}) \\ &= g(\frac{-\dot{x}y + x\dot{y}}{x^2+y^2})(\frac{r^2-z^2}{r^2+zr}) \\ &= g(1-\frac{z}{r})(\frac{x\dot{y}-\dot{x}y}{x^2+y^2}) \end{aligned} \tag{6.48}$$

In general case for both south and north, it is

$$\mathbf{A}^d.\dot{\mathbf{x}} = \frac{1}{2}(\pm 1 - z/r)\frac{x\dot{y} - y\dot{x}}{x^2+y^2}. \tag{6.49}$$

We obtain the physical potential energy for the position $\mathbf{x}$ by eliminating the auxiliary field D from the Lagrangian. Substituting (6.2) which is now $D_* = \frac{1}{m}(\frac{\kappa}{2|\mathbf{x}|} + \theta)$ into the Lagrangian, one finds

$$L(\mathbf{x}, \lambda, \bar{\lambda}) = \frac{m}{2}(\dot{\mathbf{x}}^2 + 2i\bar{\lambda}\dot{\lambda}) - \frac{1}{2m}(\frac{\kappa}{2|\mathbf{x}|} + \theta)^2 - \kappa\mathbf{A}^d.\dot{\mathbf{x}} - \frac{\kappa\,\mathbf{x}}{2|\mathbf{x}|^3}.\bar{\lambda}\boldsymbol{\sigma}\lambda \tag{6.50}$$

Potential of the system is

$$V(\mathbf{x}, \lambda) = \frac{1}{2m}(\frac{\kappa}{2|\mathbf{x}|} + \theta)^2 + \frac{\kappa\,\mathbf{x}}{2|\mathbf{x}|^3}.\bar{\lambda}\boldsymbol{\sigma}\lambda. \tag{6.51}$$

6.3. Supersymmetry Preserving Solutions

By symmetry we mean that under supersymmetric variations the Lagrangian and action of the system remain the same. Equations of motion are invariant. However, solutions to equations of motion are not necessarily invariant. [1] When the effective

[1] Consider the Lagrangian of a free particle $L = m\dot{x}^2/2$. Both the Lagrangian and the equation of motion $\ddot{x} = 0$ are invariant under rotations. i.e $x = ct$ is a solution to equation of motion but breaks the rotational symmetry. However, $\dot{x} = 0$ is another solution and satisfies the ground state energy with two possible value either $x = 0$ or $x = x_0$ which is a constant . The first solution preserves the symmetry while the second one breaks the symmetry spontaneously.

potential at the minimum is equal to zero, supersymmetry is preserved. If the Minimum of the Potential is different from zero, then the ground state is not supersymmetric and supersymmetry has been broken spontaneously. Ground state energy should be zero in the case of supersymmetry. When we consider supersymmetric transformations , in the simplest way a supersymmetric configuration is satisfied with a time independent $\mathbf{x}$ with $D = 0$ and all fermionic variables are zero $\lambda = \bar{\lambda} = 0$. That makes variations $\delta A = \delta\lambda = \delta D = \delta x$ totally zero. Also, when we apply those conditions on the Lagrangian, we immediately see it is zero. For the variation in (6.26)

$$\delta\lambda = (\dot{x}_i\sigma_i + iD)\xi = 0,$$

let's say M is not an invertible matrix with zero eigenvalues and ξ is the corresponding eigenvector to be determined.

$$M = \dot{x}_i\sigma_i + iD = \begin{pmatrix} \dot{x_3} + iD & \dot{x_1} - i\dot{x_2} \\ \dot{x_1} + i\dot{x_2} & -\dot{x_3} + iD \end{pmatrix}. \tag{6.52}$$

Determinant of the matrix vanishes and gives us condition on D and $\mathbf{x}$

$$\det M = 0 \Rightarrow D = \pm i\dot{x}_i. \tag{6.53}$$

For real solutions $D = 0$ and $\dot{x}_i = 0$. Therefore, ξ is arbitrary. If supersymmetric ground states energy $Q|\psi\rangle = 0$ where $Q = Q(\delta x, \delta D, \delta\lambda)$, solutions to equation of motion are $\dot{x} = 0, D = 0, \lambda = 0$. Even though $\delta\mathbf{x} = \delta D = 0$ are zero, $\delta\lambda$ does not have to be zero, necessarily.

$$\delta\lambda = \dot{\mathbf{x}}.\boldsymbol{\sigma}\xi + iD\xi \neq 0$$

So we wouldn't have a supersymmetry conservation and got a spontaneous supersymmetry breaking.

6.4. Interactions Between Two D-Particles

For two interacted particle the general form of the Lagrangian which is followed from [19] is given as

$$L = \sum_{i=1}^{2} \frac{m_i}{2}(\dot{\mathbf{x}}_i{}^2 + D_i^2 + 2i\dot{\lambda}_i\bar{\lambda}_i) - U_i D_i + \mathbf{A}_i.\dot{\mathbf{x}}_i + \sum_{i,j=1}^{2}(C_{ij}\bar{\lambda}_i\lambda_j + \mathbf{C}_{ij}.\bar{\lambda}_i\boldsymbol{\sigma}\lambda_j) \quad (6.54)$$

We use the center of mass terms for the first expression in the parenthesis

$$\mathbf{x}_0 = \frac{\mathbf{x}_1 m_1 + \mathbf{x}_2 m_2}{m_1 + m_2}, \qquad D_0 = \frac{m_1 D_1 + m_2 D_2}{m_1 + m_2}, \qquad \lambda_0 = \frac{m_1\lambda_1 + m_2\lambda_2}{m_1 + m_2} \quad (6.55)$$

and plug them in the Lagrangian. We get center of mass term

$$L_0 = \frac{m_1 + m_2}{2}(\dot{\mathbf{x}}_0^2 + D_0^2 + 2i\dot{\lambda}_0\bar{\lambda}_0). \quad (6.56)$$

Now the remaining part of the Lagrangian is

$$\sum_{i,j}\frac{\mu}{2}(\dot{\mathbf{r}}_{ij}^2 + D_{ij}^2 + 2i\dot{\lambda}_{ij}\bar{\lambda}_{ij}) + \sum_i(-U_i D_i + \mathbf{A}_i.\dot{\mathbf{x}}_i) + \sum_{i,j}(C_{ij}\bar{\lambda}_i\lambda_j + \mathbf{C}_{ij}.\bar{\lambda}_i\boldsymbol{\sigma}\lambda_j). \quad (6.57)$$

We add an index i=1,2 to the supersymmetry transformations and now supersymmetry conditions on the the Lagrangian become

$$\mathbf{C}_{ij} = \nabla_i U_j = \nabla_j U_i = \frac{1}{2}(\nabla_i \times \mathbf{A}_j + \nabla_j \times \mathbf{A}_i); \qquad C_{ij} = 0 \quad (6.58)$$

where

$$U_i = \sum_j \frac{\kappa_{ij}}{2r_{ij}} + \theta_i \quad (6.59)$$

with $\kappa_{ij} = -\kappa_{ji}$, and $\mathbf{A}_i$ is the vector potential produced at x_i by a magnetic monopole with charge $\{\kappa_{ij}\}_j$ at x_j. There is a singularity when two particles coincide $r_{ij} = 0$. We can write the second and third parenthesis in the Lagrangian in terms of interaction :

$$L^{(2)} = -\sum_i \theta_i D_i - \sum_{j>i} \kappa_{ij} L_{ij}^{int} \tag{6.60}$$

where the interaction term is

$$L_{ij}^{int} = \frac{1}{2r_{ij}} D_{ij} + \mathbf{A}^d(\mathbf{r}_{ij}).\dot{\mathbf{r}}_{ij} + \frac{1}{2r_{ij}^3} \mathbf{r}_{ij}.\bar{\lambda}_{ij} \boldsymbol{\sigma} \lambda_{ij} \tag{6.61}$$

where $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, $D_{ij} = D_i - D_j$, $\lambda_{ij} = \lambda_i - \lambda_j$ $\kappa_{ij} \in \mathrm{Z}$ satisfies the Dirac quantization condition (B.23). The supersymmetric minima of the interaction potential between the particles are at positions $\sum_i \kappa_{ij}/(2r_{ij}) = -\theta_i$

7. SUPERSYMMETRIC QUIVER MECHANICS

[2] The word Quiver, is used in mathematics and physics for diagrams with a number of nodes and arrows connecting the nodes. The arrows represent maps between vector spaces associated to the nodes and nodes correspond to D branes. The intersection of branes gives rise to particles. The Lagrangian which describes the classical system of D-particles connected to each other by light strings is given in [19]

$$\begin{aligned} L &= \frac{\mu}{2}(\dot{\mathbf{x}}^2 + D^2 + 2i\bar{\lambda}\dot{\lambda}) - \theta D + \mid \mathrm{D}_t\phi \mid^2 - (\mathbf{x}^2 + D) \mid \phi \mid^2 + \mid F \mid^2 + i\bar{\psi}\mathrm{D}_t\psi \\ & -\bar{\psi}\mathbf{x}.\boldsymbol{\sigma}\psi - i\sqrt{2}(\bar{\phi}\psi\epsilon\lambda - \bar{\lambda}\epsilon\bar{\psi}\phi) \end{aligned} \tag{7.1}$$

with $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$, $\mu = m_1 m_2/(m_1 + m_2)$ and θ is a constant. $\mathrm{D}_t\phi = \partial_t\phi + iA\phi$ is the covariant derivative. A is a non dynamical one dimensional gauge potential and appears only in covariant derivative. $(\mathbf{x}, A, D, \lambda)$ are vector multiplet and (ϕ, F, ψ) are chiral multiplet where ϕ is complex scalar, F is an auxiliary complex scalar and ψ is a 2 component spinor [19]. Equations of motion for the dynamical vector multiplet are:

$$\ddot{x}_i + \frac{2}{\mu} \mid \phi \mid^2 x_i + \frac{1}{\mu}\bar{\psi}\sigma_i\psi = 0 \tag{7.2}$$

$$\dot{\lambda} + \frac{\sqrt{2}}{\mu}\epsilon\bar{\psi}\phi = 0 \tag{7.3}$$

For the chiral multiplet:

$$\mathrm{D}_t\mathrm{D}_t\phi + (x^2 + D)\phi + i\sqrt{2}\psi\epsilon\lambda = 0 \tag{7.4}$$

$$\mathrm{D}_t\psi + ix_i\sigma_i\psi - \sqrt{2}\bar{\lambda}\epsilon\phi = 0. \tag{7.5}$$

For the non-dynamical fields:

$$D = \frac{\theta + \mid \phi \mid^2}{\mu} \tag{7.6}$$

[2]Streched strings leads to charged chiral multiplets at the intersection points of branes. The sign of the charge depend on the sign of the intersection.

F is an auxiliary field which is $\bar{F} = F = 0$ but it seems in the Lagrangian.

$$A = \frac{\mid \psi \mid^2 + i(\bar{\phi}\dot{\phi} - \dot{\bar{\phi}}\phi)}{2\phi\bar{\phi}} \tag{7.7}$$

In addition, we have one more equation that is equivalent to (7.7)

$$\bar{\phi} D_t \phi - (\bar{D}_t \bar{\phi})\phi - i\bar{\psi}\psi = 0. \tag{7.8}$$

7.1. Potential And Minima

We eliminate the auxiliary fields D and F from the Lagrangian

$$\begin{aligned} L &= \frac{\mu}{2}(\dot{\mathbf{x}}^2 + \frac{(\theta + \mid \phi \mid^2)^2}{\mu^2} + 2i\bar{\lambda}\dot{\lambda}) - \theta\frac{(\theta + \mid \phi \mid^2)}{\mu} + \mid \mathrm{D}_t \phi \mid^2 - (\mathbf{x}^2 + \frac{(\theta + \mid \phi \mid^2)}{\mu}) \mid \phi \mid^2 \\ &\quad + i\bar{\psi}\mathrm{D}_t\psi - \bar{\psi}\mathbf{x}.\boldsymbol{\sigma}\psi - i\sqrt{2}(\bar{\phi}\psi\epsilon\lambda - \bar{\lambda}\epsilon\bar{\psi}\phi) \\ &= \frac{\mu}{2}(\dot{\mathbf{x}}^2 + 2i\bar{\lambda}\dot{\lambda}) - \frac{(\theta + \mid \phi \mid^2)^2}{2\mu} - x^2 \mid \phi \mid^2 + i\bar{\psi}\mathrm{D}_t\psi - \bar{\psi}\mathbf{x}.\boldsymbol{\sigma}\psi \\ &\quad - i\sqrt{2}(\bar{\phi}\psi\epsilon\lambda - \bar{\lambda}\epsilon\bar{\psi}\phi) \end{aligned} \tag{7.9}$$

where the potential term is

$$V(\phi, x) = \frac{(\theta + \mid \phi \mid^2)^2}{2\mu} + x^2 \mid \phi \mid^2 \tag{7.10}$$

$$= \frac{\theta^2}{2\mu} + m_\phi^2 \mid \phi \mid^2 + \frac{\mid \phi \mid^4}{2\mu} \tag{7.11}$$

where $m_\phi^2 = \mathbf{x}^2 + \theta/\mu$ is the mass of the ϕ- modes like the stretched bosonic string masses. When $\phi = 0$ and $x = 0$ which means disconnected branes, the potential energy is $\theta^2/(2\mu)$. Classical ground states of this system can be determined from the local minima of potential:

$$\frac{\partial V}{\partial x} = 2x \mid \phi \mid^2 = 0 \Rightarrow x = 0 \quad \text{or} \quad \mid \phi \mid^2 = 0 \tag{7.12}$$

$$\frac{\partial V}{\partial \phi} = \bar{\phi}(x^2 + \frac{1}{\mu}(\theta + \mid \phi \mid^2)) = 0 \Rightarrow \bar{\phi} = 0 \quad \text{or} \quad x^2 = -\frac{1}{\mu}(\theta + \mid \phi \mid^2). \tag{7.13}$$

If $\theta < 0$ there are two branches :

1. The coulomb branch: $\phi = 0$
2. The higgs branch: $x = 0, \quad \theta = - \mid \phi \mid^2$

Stable and unstable conditions are determined by the sign of the second derivations of the potential:

$$\begin{pmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial \phi} & \frac{\partial^2 V}{\partial x \partial \bar{\phi}} \\ \frac{\partial^2 V}{\partial \phi \partial x} & \frac{\partial^2 V}{\partial \phi \partial \phi} & \frac{\partial^2 V}{\partial \phi \partial \bar{\phi}} \\ \frac{\partial^2 V}{\partial \bar{\phi} \partial x} & \frac{\partial^2 V}{\partial \bar{\phi} \partial \phi} & \frac{\partial^2 V}{\partial \bar{\phi} \partial \bar{\phi}} \end{pmatrix} = \begin{pmatrix} 2\bar{\phi}\phi & 2x\bar{\phi} & 2x\phi \\ 2x\bar{\phi} & \frac{1}{\mu}\bar{\phi}^2 & \frac{1}{\mu}(\theta + 2 \mid \phi \mid^2) + x^2 \\ 2x\phi & \frac{1}{\mu}(\theta + 2 \mid \phi \mid^2) + x^2 & \frac{1}{\mu}\phi^2 \end{pmatrix}$$

Consider the case $\phi = 0$ which is the Coulomb branch condition. The only non zero term is $\frac{\partial^2 V}{\partial \bar{\phi} \partial \phi} = \theta/\mu + x^2$. For $x^2 > -\theta/\mu$ potential is stable. Otherwise, $x^2 < -\theta/\mu$ and $x = 0$ yield unstable configuration. At the point $x^2 = -\theta/\mu$ potential gets flat. Another case with $x = 0$ and $\phi \neq 0$ conditions give either flat or stable potentials. This corresponds to the Higgs branch case.

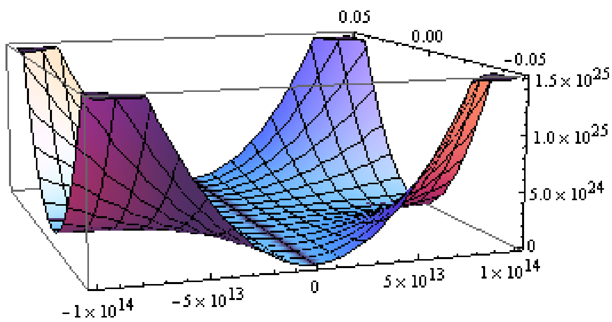

Figure 7.1. V(ϕ, x): The potential is unstable around the point $x = 0$ and $\phi = 0$ that obeys the coulomb branch condition. Also, at $x = 0$ one notes that the potential is stable at two points of $\phi \sim \pm\sqrt{-\theta}$ which satisfy the Higgs branch condition. Increasing x at $\phi = 0$ leads to stable potential.

The figures below tell us the behaviour of the potential along the separate directions when we keep ϕ or x constant while the other is changing. A ground state is supersymmetric if the potential is zero. Therefore, at the classical level only the Higgs branch satisfies the supersymmetric ground states. This is true for $\theta < 0$. If θ is zero, coulomb branch provides supersymmetric ground states.

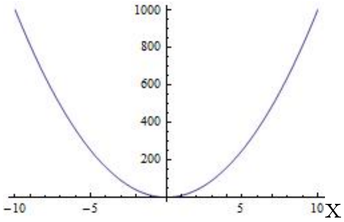

Figure 7.2. $V(x)$: Potential depends on only x when $\phi \neq 0$ is a constant. $x = 0$ is the stable point.

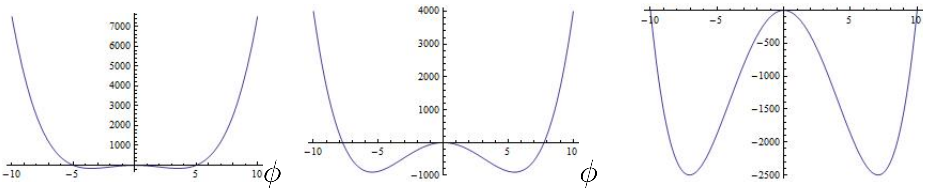

Figure 7.3. $V(\phi)$: For $x = 0$, $\phi = 0$ is the unstable point. However, potential is stable at two points that correspond to the Higgs branch condition.

7.2. Supersymmetry Transformation

We separate the Lagrangian into vector multiplet, chiral multiplet and interaction parts:

$$L = L_{V,FI} + L_C + L_{int} \tag{7.14}$$

Corresponding supersymmetry transformations are

$$\begin{aligned}
\delta A &= i\bar{\lambda}\xi - i\bar{\xi}\lambda && (7.15)\\
\delta x &= i\bar{\lambda}\boldsymbol{\sigma}\xi - i\bar{\xi}\boldsymbol{\sigma}\lambda && (7.16)\\
\delta\lambda &= \dot{\mathbf{x}}.\boldsymbol{\sigma}\xi + iD\xi && (7.17)\\
\delta D &= -\dot{\bar{\lambda}}\xi - \bar{\xi}\dot{\lambda} && (7.18)\\
\delta\phi &= \sqrt{2}\epsilon\xi\psi && (7.19)\\
\delta\psi &= -i\sqrt{2}\bar{\xi}\epsilon D_t\phi - \sqrt{2}\mathbf{x}.\boldsymbol{\sigma}\bar{\xi}\epsilon\phi + \sqrt{2}\xi F && (7.20)\\
\delta F &= -i\sqrt{2}\bar{\xi}\epsilon D_t\psi + \sqrt{2}\bar{\xi}\boldsymbol{\sigma}\psi.\mathbf{x} - 2i\bar{\xi}\epsilon\bar{\lambda}\phi && (7.21)
\end{aligned}$$

Lagrangian with vector multiplets is

$$L_{V,FI} = \frac{\mu}{2}(\dot{x}^2 + D^2 + 2i\bar{\lambda}\dot{\lambda}) - \theta D. \tag{7.22}$$

Supersymmetric transformations on the Lagrangian will give a total derivation

$$\begin{aligned}
\delta L_{V,FI} &= \mu(\dot{x}\delta\dot{x} + D\delta D + i\delta\bar{\lambda}\dot{\lambda} + i\bar{\lambda}\delta\dot{\lambda}) - \theta\delta D\\
&= \mu(i\dot{x}(\dot{\bar{\lambda}}\sigma\xi - \bar{\xi}\sigma\dot{\lambda}) - D(\dot{\bar{\lambda}}\xi + \bar{\xi}\dot{\lambda}) + i(\dot{x}_i\bar{\xi}\sigma_i - iD\bar{\xi})\dot{\lambda} + i\bar{\lambda}(\ddot{x}_i\sigma_i\xi + i\dot{D}\xi))\\
&\quad +\theta(\dot{\bar{\lambda}}\xi + \bar{\xi}\dot{\lambda})\\
&= \frac{d}{dt}\bar{\lambda}(i\dot{x}_i\sigma_i - D + \frac{\theta}{\mu})\xi + \frac{d}{dt}(\frac{\theta}{\mu})\bar{\xi}\lambda
\end{aligned} \tag{7.23}$$

In a similar way, Lagrangian with chiral multiplet is

$$L_C = \mid \dot{\phi} \mid^2 + \mid F \mid^2 + i\bar{\psi}\dot{\psi}. \tag{7.24}$$

The variation is

$$\delta L_C = \delta\dot{\bar{\phi}}\dot{\phi} + \dot{\bar{\phi}}\delta\dot{\phi} + \delta\bar{F}F + \bar{F}\delta F + i\delta\bar{\psi}\dot{\psi} + i\bar{\psi}\delta\dot{\psi}. \tag{7.25}$$

We use the supersymmetry transformations (7.20) and (7.21) which do not contain interaction terms in the variation of the Lagrangian.

$$\begin{aligned}\delta L_C &= \sqrt{2}\dot{\bar{\psi}}\bar{\xi}\epsilon\dot{\phi} + \sqrt{2}\dot{\bar{\phi}}\epsilon\xi\dot{\psi} + i\sqrt{2}\dot{\bar{\psi}}\xi F - i\sqrt{2}\bar{F}\bar{\xi}\dot{\psi} - \sqrt{2}\dot{\bar{\phi}}\epsilon\xi\dot{\psi} \\ &\quad + i\sqrt{2}\bar{F}\bar{\xi}\dot{\psi} + \sqrt{2}\bar{\psi}\bar{\xi}\epsilon\ddot{\phi} + i\sqrt{2}\bar{\psi}\xi\dot{F} \\ &= \sqrt{2}\frac{d}{dt}(\bar{\psi}\dot{\phi}\bar{\xi}\epsilon + i\bar{\psi}\xi F)\end{aligned} \tag{7.26}$$

We stopped here and didn't go further. We leave the check of supersymmetry for the interaction part to people who are interested in.

7.3. Supersymmetry Preserving Solutions

For bosonic solutions all fermionic variables are $\lambda = \psi = 0$. Now Lagrangian is

$$L = \frac{\mu}{2}\dot{x}^2 + \frac{1}{2\mu}(\mid \phi \mid^2 + \theta)^2 - \theta\frac{1}{\mu}(\mid \phi \mid^2 + \theta) - (x^2 + \frac{1}{\mu}(\mid \phi \mid^2 + \theta)) \mid \phi \mid^2 + \mid F \mid^2 + \mid D_t\phi \mid^2 \tag{7.27}$$

Equations of motion become

$$\ddot{x} + \frac{2}{\mu}|\phi|^2 x = 0 \tag{7.28}$$

$$D_t D_t\phi + (x^2 + D)\phi = 0. \tag{7.29}$$

The variation of bosonic variables are automatically zero when we put fermionic variables to zero $\delta A = \delta x = \delta D = \delta\phi = 0$. We need some conditions on fermionic transformations to make them zero

$$\delta\lambda = (\dot{x}_i\sigma_i + iD)\xi = 0.$$

This is the same argument as we did before in section 6.3

$$\dot{x}_i = 0, \qquad D = \frac{\theta + \mid \phi \mid^2}{\mu} = 0. \tag{7.30}$$

Special solution is

$$x = 0, \qquad \theta = - \mid \phi \mid^2 \tag{7.31}$$

which is in agreement with Higgs Branch. Another fermionic transformation (7.20) should also be zero:

$$\delta\psi = -i\sqrt{2}\bar{\xi}\epsilon D_t\phi - \sqrt{2}x_i\sigma_i\bar{\xi}\epsilon\phi + \sqrt{2}\xi F = 0 \tag{7.32}$$

We denote the matrix $\tilde{M} = 1D_t\phi - ix_i\sigma_i\phi$. Explicitly,

$$\tilde{M} = \begin{pmatrix} (D_t - ix_3)\phi & -i\bar{z}\phi \\ -iz\phi & (D_t + ix_3)\phi \end{pmatrix} \tag{7.33}$$

where $z = x_1 + ix_2$. Now, the equation (7.32) is

$$\sqrt{2}\tilde{M}\bar{\xi}\epsilon + i\sqrt{2}\xi F = 0. \tag{7.34}$$

The eigenvectors are :

$$\xi_a = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \qquad \text{and} \qquad (\bar{\xi}\epsilon)_\alpha = \bar{\xi}^\beta\epsilon_{\beta\alpha} = \begin{pmatrix} \bar{\xi}_2 \\ -\bar{\xi}_1 \end{pmatrix} \tag{7.35}$$

The matrix equation (7.34) is

$$\begin{pmatrix} (D_t - ix_3)\phi & -i\bar{z}\phi \\ -iz\phi & (D_t + ix_3)\phi \end{pmatrix} \begin{pmatrix} \bar{\xi_2} \\ -\bar{\xi_1} \end{pmatrix} + i \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0. \tag{7.36}$$

We get four equations for ξ_1, ξ_2 and $\bar{\xi_1}$, $\bar{\xi_2}$ as follows

$$(D_t - ix_3)\phi\bar{\xi_2} + i\bar{z}\phi\bar{\xi_1} + iF\xi_1 = 0 \tag{7.37}$$

$$-iz\phi\bar{\xi_2} - (D_t + ix_3)\phi\bar{\xi_1} + iF\xi_2 = 0 \tag{7.38}$$

$$(\bar{D}_t + ix_3)\bar{\phi}\xi_2 - iz\bar{\phi}\xi_1 - i\bar{F}\bar{\xi_1} = 0 \tag{7.39}$$

$$i\bar{z}\bar{\phi}\xi_2 - (\bar{D}_t - ix_3)\bar{\phi}\xi_1 - i\bar{F}\bar{\xi_2} = 0 \tag{7.40}$$

They form a 4×4 matrix equation:

$$\begin{pmatrix} iF & i\bar{z}\phi & 0 & (D_t - ix_3)\phi \\ 0 & -(D_t + ix_3)\phi & iF & -iz\phi \\ -iz\bar{\phi} & -i\bar{F} & (\bar{D}_t + ix_3)\bar{\phi} & 0 \\ -(\bar{D}_t - ix_3)\bar{\phi} & 0 & i\bar{z}\bar{\phi} & -i\bar{F} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \bar{\xi_1} \\ \xi_2 \\ \bar{\xi_2} \end{pmatrix} = 0$$

Determinant of the matrix is zero when we take $F = 0$ and $D_t\phi = ix_3\phi = 0$ and $x = 0$. ξ is arbitrary. The results derived from the matrix is consistent with the Higgs branch. In the Higgs branch formalism potential and equations of motion are

$$V = 0 \tag{7.41}$$

$$\ddot{x} = 0, \quad D = 0 \tag{7.42}$$

$$D_t D_t \phi = 0 \tag{7.43}$$

It is compatible with the equations of motion and preserve the supersymmetry in the ground state.

8. SEPARATION OF SCALES

In [19] Quiver quantum mechanics involves both chiral and vector multiplets. If we separate the branes enough, chiral multiplets become massive and since the Lagrangian (7.1) is quadratic, chiral multiplets can be integrated out. Then, the effective bosonic Lagrangian is given in the following form:

$$L_{eff} = \frac{\mu}{2}(\dot{\mathbf{x}}^2 + D^2) - \theta D - \kappa\sqrt{x^2 + D} + \kappa x \tag{8.1}$$

where $x = |\mathbf{x}|$ and the masses of the chiral mode ϕ and fermions ψ are $\sqrt{x^2 + D}$ and x,respectively. Assume $x \gg D$. Then, the Lagrangian is written in the following form:

$$\begin{aligned} L_{eff} &= \frac{\mu}{2}(\dot{\mathbf{x}}^2 + D^2) - \theta D - \kappa x(1 + \frac{D}{2x^2}) + \kappa x && (8.2) \\ &= \frac{\mu}{2}(\dot{\mathbf{x}}^2 + D^2) - D(\theta + \frac{\kappa}{2x}) && (8.3) \end{aligned}$$

It is similar to the bosonic Lagrangian with a Dirac magnetic monopole in section (6.46). Insert D into (8.3):

$$D = \frac{1}{\mu}(\theta + \frac{\kappa}{2x}) \tag{8.4}$$

$$L_{eff} = \frac{\mu}{2}\dot{\mathbf{x}}^2 - \frac{1}{2\mu}(\theta + \frac{\kappa}{2x})^2 \tag{8.5}$$

The minimum of the potential is $x = -\kappa/2\theta$ and it is stable solution (see fig:8.1). In our model we address the question whether classical approximations give the same effective Lagrangian (8.1) in quantum case or not. Classically , one way is to separate the scales. The time scale of the chiral multiplet is much smaller than the vector multiplet. We determine fast and small oscillating fields in the Lagrangian and eliminate the time dependent effect of fast variables by taking the average of them in a long period and obtain effective potentials which give the equation of motion for slowly changing field.

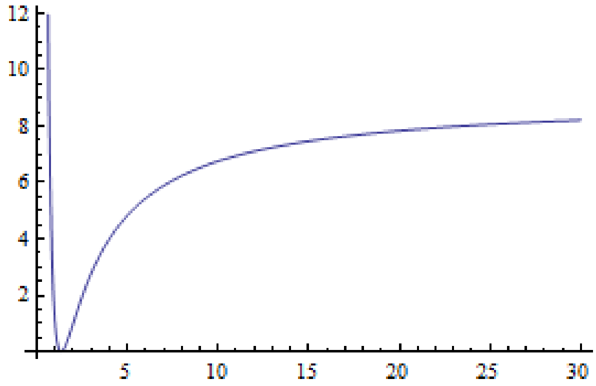

Figure 8.1. V(x): $\lim_{x\to 0}$, potential diverges and $\lim_{x\to +\infty}$, it gets a constant value.

8.1. Fast Perturbations of an Oscillating Field

In this section we closely follow the discussion in [20], [21] to understand how to deal with the fast oscillating fields in our model. Here the force which causes to the rapidly oscillation is external. In our model there is no external force acting on the system. Also, we assume there is no certain fast and slow field. The interacting fields behave as fast and slow oscillator relative to each other. To have a general idea, let's have a look at the behaviour of a particle in an external force with a high frequency. Consider the motion of a particle in a time independent field of potential U. An external force is applied on the particle:

$$F(t) = F \sin wt \tag{8.6}$$

which varies in time with a high frequency w that is bigger than the natural frequency $w \gg w_0 = 2\pi/T_0$ that is when the particle is only in the U field alone

$$q = q_h(t) - \frac{F}{mw^2} \sin wt. \tag{8.7}$$

The amplitude of the oscillation

$$\delta q = -\frac{F}{mw^2} \sin wt \tag{8.8}$$

is small due to high frequency. Consider a general system with

$$H(q,p,t) = H_0(q,p) + V(q)sin(wt+\delta). \tag{8.9}$$

Due to the field the particle will move across a slow path and at the same time execute small but rapid oscillations along the path. We look for a solution that involves both slow and fast components such as

$$\begin{aligned} q(t) &= \bar{q}(t) + \xi(t) && (8.10)\\ p(t) &= \bar{p}(t) + \pi(t) && (8.11) \end{aligned}$$

where $\bar{q}(t)$, $\bar{p}(t)$ are the slow and $\xi(t)$,$\pi(t)$ correspond to fast part of the motion. Now, we expand the Hamilton's equations in these quantities:

$$\begin{aligned} \dot{\bar{q}} + \dot{\xi} &= \frac{\partial H_0}{\partial \bar{p}} + \frac{\partial^2 H_0}{\partial \bar{p}^2}\pi + \frac{\partial^2 H_0}{\partial \bar{q}\partial \bar{p}}\xi + \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q}^2 \partial \bar{p}}\xi^2 + \frac{\partial^3 H_0}{\partial \bar{q}\partial \bar{p}^2}\xi\pi + \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{p}^3}\pi^2 && (8.12)\\ \dot{\bar{p}} + \dot{\pi} &= -\frac{\partial H_0}{\partial \bar{q}} - \frac{\partial^2 H_0}{\partial \bar{q}^2}\xi - \frac{\partial^2 H_0}{\partial \bar{q}\partial \bar{p}}\pi - \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q}^3}\xi^2 - \frac{\partial^3 H_0}{\partial \bar{q}^2\partial \bar{p}}\xi\pi - \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q}\partial \bar{p}^2}\pi^2 \\ &\quad -\frac{\partial V}{\partial \bar{q}}sin(wt+\delta) - \frac{\partial^2 V}{\partial \bar{q}^2}\xi sin(wt+\delta) && (8.13) \end{aligned}$$

In a long interval of time let us compute the average of fluctuating quantities in (8.12) and (8.13). The mean value $\frac{1}{T}\int_0^T ...dt$ of $\xi(t)$ over the period $T = \frac{2\pi}{w}$ is zero. $\bar{q}(t)$ changes only slightly in that time therefore it describes the slow motion of the particle averaged over the rapid oscillations. The mean values of the first power of ξ and π are zero since they describe the sinusoidal motion

$$< \xi >=< \pi >= 0. \tag{8.14}$$

The next non zero terms in averages are the second power of ξ and π. Therefore, the lowest non trivial order in averages are

$$\begin{aligned} \dot{\bar{q}} &= \frac{\partial H_0}{\partial \bar{p}} + \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q}^2 \partial \bar{p}} < \xi^2 > + \frac{\partial^3 H_0}{\partial \bar{q} \partial \bar{p}^2} < \xi\pi > + \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{p}^3} < \pi^2 > \qquad (8.15) \\ \dot{\bar{p}} &= -\frac{\partial H_0}{\partial \bar{q}} - \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q}^3} < \xi^2 > - \frac{\partial^3 H_0}{\partial \bar{q}^2 \partial \bar{p}} < \xi\pi > - \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q} \partial \bar{p}^2} < \pi^2 > \\ & \quad - \frac{\partial^2 V}{\partial \bar{q}^2} < \xi sin(wt+\delta) > \qquad (8.16) \end{aligned}$$

The first degrees of freedom:

$$\begin{aligned} \dot{\xi} &= \frac{\partial^2 H_0}{\partial \bar{q} \partial \bar{p}}\xi + \frac{\partial^2 H_0}{\partial \bar{p}^2}\pi \qquad (8.17) \\ &= A\xi + B\pi \qquad (8.18) \\ \dot{\pi} &= -\frac{\partial^2 H_0}{\partial \bar{q}^2}\xi - \frac{\partial^2 H_0}{\partial \bar{q} \partial \bar{p}}\pi - \frac{\partial V}{\partial \bar{q}} sin(wt+\delta) \qquad (8.19) \\ &= -C\xi - A\pi + Fe^{-iwt} \qquad (8.20) \end{aligned}$$

For simplicity we redefine the equations and write general solutions in the form of

$$\begin{pmatrix} \xi \\ \pi \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{-iwt}. \qquad (8.21)$$

We have two equations to be solved easily:

$$\begin{aligned} iw\alpha &= -A\alpha - B\beta \qquad (8.22) \\ iw\beta &= C\alpha + A\beta - F \qquad (8.23) \end{aligned}$$

with the coefficients

$$\begin{aligned} \alpha &= \frac{BF}{BC - A^2 - w^2} = -\frac{BF}{w^2} + O(w^{-4}) \qquad (8.24) \\ \beta &= -\frac{(A+iw)F}{BC - A^2 - w^2} = \frac{iF}{w} + O(w^{-3}). \qquad (8.25) \end{aligned}$$

Taking the real part of the solutions with a phase shift δ

$$\xi(t) = -\frac{BF}{w^2}sin(wt+\delta) = \frac{1}{w^2}\frac{\partial V}{\partial \bar{q}}\frac{\partial^2 H_0}{\partial \bar{p}^2}sin(wt+\delta) \tag{8.26}$$

$$\pi(t) = -\frac{F}{w}cos(wt+\delta) = \frac{1}{w}\frac{\partial V}{\partial \bar{q}}cos(wt+\delta) \tag{8.27}$$

The averages to the lowest order are:

$$<\xi^2> = \frac{1}{2w^4}(\frac{\partial V}{\partial \bar{q}})^2(\frac{\partial^2 H_0}{\partial \bar{p}^2})^2 \tag{8.28}$$

$$<\pi^2> = \frac{1}{2w^2}(\frac{\partial V}{\partial \bar{q}})^2 \tag{8.29}$$

$$<\xi sin(wt+\delta)> = \frac{1}{2w^2}\frac{\partial V}{\partial \bar{q}}\frac{\partial^2 H_0}{\partial \bar{p}^2} \tag{8.30}$$

$$<\xi\pi> = 0. \tag{8.31}$$

Substitute the averages into the equation of motion for the slow variables $\bar{q}$ and $\bar{p}$:

$$\dot{\bar{q}} = \frac{\partial H_0}{\partial \bar{p}} + \frac{1}{4w^4}\frac{\partial^3 H_0}{\partial \bar{q}^2 \partial \bar{p}}(\frac{\partial V}{\partial \bar{q}})^2(\frac{\partial^2 H_0}{\partial \bar{p}^2})^2 + \frac{1}{4w^2}\frac{\partial^3 H_0}{\partial \bar{p}^3}(\frac{\partial V}{\partial \bar{q}})^2 \tag{8.32}$$

$$\begin{aligned}\dot{\bar{p}} = & -\frac{\partial H_0}{\partial \bar{q}} - \frac{1}{4w^4}\frac{\partial^3 H_0}{\partial \bar{q}^3}(\frac{\partial V}{\partial \bar{q}})^2(\frac{\partial^2 H_0}{\partial \bar{p}^2})^2 - \frac{1}{4w^2}\frac{\partial^3 H_0}{\partial \bar{q}\partial \bar{p}^2}(\frac{\partial V}{\partial \bar{q}})^2 \\ & -\frac{1}{2w^2}\frac{\partial^2 V}{\partial \bar{q}^2}\frac{\partial V}{\partial \bar{q}}\frac{\partial^2 H_0}{\partial \bar{p}^2}.\end{aligned} \tag{8.33}$$

We obtain a new Hamiltonian independent of time

$$\tilde{H}(\bar{q},\bar{p}) = H_0(\bar{q},\bar{p}) + \frac{1}{4w^2}(\frac{\partial^2 H_0}{\partial \bar{p}^2})(\frac{\partial V}{\partial \bar{q}})^2. \tag{8.34}$$

which gives the same equation of motion for slowly oscillating fields. Thus the motion of the particle averaged over the oscillations is the same as if the constant potential V was increased by a constant quantity proportional to the squared amplitude of the variable field. The important point here is that by using this method we get effective potentials which are not dependent on time.

8.2. Separation of Scales for Two Interacting Scalar Fields

For our model consider the Lagrangian is

$$L = \frac{\mu}{2}\dot{\mathbf{x}}^2 + \dot{\bar{\phi}}\dot{\phi} - (\mathbf{x}^2 + D)\bar{\phi}\phi. \tag{8.35}$$

where $\mathbf{x}$ and D are slow oscillating fields and remain effectively constant in a long period. ϕ is the fast oscillating field. Equations of motion are:

$$\ddot{x}_i + \frac{2x_i}{\mu}\bar{\phi}\phi = 0 \tag{8.36}$$

$$\ddot{\phi} + (\mathbf{x}^2 + D)\phi = 0. \tag{8.37}$$

We assume the real solution for $\phi \approx A\cos(\sqrt{\mathbf{x}^2 + D}t)$ and A is the amplitude of the periodic motion to be determined. Plug ϕ in (8.36):

$$\ddot{x}_i + \frac{2x_i}{\mu}A^2\cos^2(\sqrt{\mathbf{x}^2 + D}t) = 0 \tag{8.38}$$

The average of the equation of motion over a long period:

$$\ddot{x}_i + \frac{2x_i}{\mu} < A^2\cos^2(\sqrt{\mathbf{x}^2 + D}t) >= 0 \tag{8.39}$$

with the average is $< \cos^2(\sqrt{\mathbf{x}^2 + D}t) >= 1/2$.

$$\ddot{x}_i + \frac{A^2}{\mu}x_i = 0 \tag{8.40}$$

Consider x and D are almost time independent, then adiabatic invariant (4.86) is $I = E/w$ where the frequency $w = \sqrt{\mathbf{x}^2 + D}$ is the slowly changing parameter. The energy of the system is

$$E = \dot{\phi}^2 + w^2\phi^2 = 2A^2w^2. \tag{8.41}$$

We compute the adiabatic invariant

$$I = 2A^2 w \tag{8.42}$$

is a constant. The amplitude is related to the adiabatic invariant as

$$A = \sqrt{\frac{I}{2w}}. \tag{8.43}$$

It is proportional to $\sqrt{1/w}$:

$$A \sim \frac{1}{(\mathbf{x}^2 + D)^{1/4}}. \tag{8.44}$$

Therefore, the effective potential is

$$V_{eff} = \sqrt{\mathbf{x}^2 + D}. \tag{8.45}$$

It is consistent with the effective Lagrangian (8.1).

8.3. Toy Model of A Scalar Field Coupled to A Fermionic Field

Here, scalar field is coupled to a fermionic field. For a Lagrangian depend on ψ and x

$$L_{\psi,x} = \frac{\mu}{2}\dot{\mathbf{x}}^2 + i\bar{\psi}\dot{\psi} - \bar{\psi}\mathbf{x}.\boldsymbol{\sigma}\psi \tag{8.46}$$

Equations of motion are

$$\mu\ddot{x}_i + \bar{\psi}\sigma_i\psi = 0 \tag{8.47}$$

$$\dot{\bar{\psi}} - i\bar{\psi}x_i\sigma_i = 0. \tag{8.48}$$

Now,we assume that ψ is the fast oscillating fermionic field and $\mathbf{x}$ doesn't vary during the motion. Solutions are

$$\bar{\psi} = \bar{\psi}_0 e^{i\mathbf{x}.\boldsymbol{\sigma}t}, \qquad \psi = e^{-i\mathbf{x}.\boldsymbol{\sigma}t}\psi_0 \tag{8.49}$$

where ψ_0 is the initial fermionic field. We use the identities:

$$\begin{aligned} e^{i\mathbf{x}.\boldsymbol{\sigma}t} &= \cos xt + i\hat{x}_i\sigma_i \sin xt &(8.50)\\ \sigma_i\sigma_j &= \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\sigma_k &(8.51)\\ [\sigma_i, \sigma_j] &= 2i\epsilon_{ijk}\sigma_k. &(8.52) \end{aligned}$$

Plug them in (8.47):

$$\mu\ddot{x}_i + \bar{\psi}_0 e^{i\mathbf{x}.\boldsymbol{\sigma}t}\sigma_i e^{-i\mathbf{x}.\boldsymbol{\sigma}t}\psi_0 = 0 \tag{8.53}$$

where

$$e^{i\mathbf{x}.\boldsymbol{\sigma}t}\sigma_i e^{-i\mathbf{x}.\boldsymbol{\sigma}t} = (\cos xt + i\hat{x}_j\sigma_j \sin xt)\sigma_i(\cos xt - i\hat{x}_k\sigma_k \sin xt). \tag{8.54}$$

For the calculations see the Appendix. The result is

$$\mu\ddot{x}_i + \bar{\psi}_0(\cos(2xt)\sigma_i + \sin(2xt)\epsilon_{ijk}\hat{x}_j\sigma_k + 2\sin^2(xt)\hat{x}_i\hat{x}_k\sigma_k)\psi_0 = 0. \tag{8.55}$$

Let' s define

$$M_i = \cos(2xt)\sigma_i + \sin(2xt)\epsilon_{ijk}\hat{x}_j\sigma_k + 2\sin^2(xt)\hat{x}_i\hat{x}_k\sigma_k \tag{8.56}$$

and (8.47) becomes

$$\mu\ddot{x}_i + \bar{\psi}_0 M_i\psi_0 = 0. \tag{8.57}$$

x_i behaves as a constant since it slowly changes over a long period. When we take the average of the equation,

$$\begin{aligned}\mu\ddot{x}_i + <\bar{\psi}_0 M_i \psi_0> &= \mu\ddot{x}_i + <\cos(2xt)> \bar{\psi}_0\sigma_i\psi_0 + <\sin(2xt)> \bar{\psi}_0\epsilon_{ijk}\hat{x}_j\sigma_k\psi_0 \\ &\quad +2 <\sin^2(xt)> \bar{\psi}_0\hat{x}_i\hat{x}_k\sigma_k\psi_0 = 0 \end{aligned} \tag{8.58}$$

the second and third term on the right hand side are zero, only the average of the second order sinusoidal term remains. The equation of motion becomes

$$\mu\ddot{x}_i + \hat{x}_i\bar{\psi}_0\hat{\mathbf{x}}.\boldsymbol{\sigma}\psi_0 = 0 \tag{8.59}$$

Hamiltonian of the fermionic part (8.46):

$$H = \bar{\psi}\mathbf{x}.\boldsymbol{\sigma}\psi. \tag{8.60}$$

Hamiltonian is the energy that is the constant of the motion.

$$E = x\bar{\psi}_0\hat{x}_i M_i\psi_0. \tag{8.61}$$

where $\hat{x}_i M_i = \hat{\mathbf{x}}.\boldsymbol{\sigma}.M_i$ is a time dependent matrix but when multiplied by a unit vector we realize it gives time independent term which is consistent with the constant energy.

$$\begin{aligned}\hat{x}_i M_i &= \hat{x}_i\cos(2xt)\sigma_i + 2\sin^2(xt)\hat{x}_k\sigma_k \\ &= \cos^2(xt)\hat{x}_i\sigma_i + \sin^2(xt)\hat{x}_i\sigma_i = \hat{x}_i\sigma_i. \end{aligned} \tag{8.62}$$

Remember the adiabatic invariant for the fermion system (4.95) and (4.96) is

$$I = \bar{\psi}\hat{\mathbf{x}}.\boldsymbol{\sigma}\psi. \tag{8.63}$$

We can write it as follows:

$$= \bar{\psi}_0 \hat{x}_i M_i \psi_0 \tag{8.64}$$

and it is related to (8.57) as

$$\ddot{x}_i + \frac{I}{\mu}\hat{x}_i = 0. \tag{8.65}$$

The corresponding effective potential is proportional to

$$V_{eff} = \frac{I}{\mu}|\mathbf{x}| \tag{8.66}$$

Again,we see this result is compatible with (8.1). When the distance x increases, the frequency of $\mathbf{x}$ field in (8.44) becomes

$$\sim \frac{1}{x}\sqrt{1 + D/x^2} \simeq \frac{1}{x}(1 - \frac{D}{2x^2}) \tag{8.67}$$

The assumption in the beginning of the chapter was $x \gg D$ so we get the similar frequency as (8.65). Growing distance leads to decrease in the frequency of x while the mass and the frequency of ϕ and ψ increase with the distance. In (8.1) Lagrangian involves only vector multiplets, we have ignored chiral modes since at large distances they become more massive and oscillate with higher frequency.

8.4. First Steps Towards The Full Model

Let's see the general Lagrangian (7.1) is Gauge invariant under the gauge transformations

$$A \to A' = A - \dot{\alpha}, \quad \phi \to \phi' = e^{i\alpha(t)}\phi, \quad \psi \to \psi' = e^{i\alpha(t)}\psi, \quad F \to F' = e^{i\alpha(t)}F$$

and

$$D_t\phi \to \dot{\phi}' + iA'\phi' = (i\dot{\alpha}\phi + \dot{\phi} + iA\phi - i\dot{\alpha}\phi)e^{i\alpha(t)t} = e^{i\alpha(t)t}D_t\phi$$

similarly,

$$D_t\psi \to e^{i\alpha(t)t}D_t\psi$$

Insert them in the Lagrangian, it is enough to look at the terms in $L^{(2)}$:

$$L'^{(2)} = D_t\phi'\bar{D}_t\bar{\phi}' - \omega^2\bar{\phi}'\phi' + \bar{F}'F' + i\bar{\psi}'D_t\psi' - \bar{\psi}'\dot{\mathbf{x}}.\boldsymbol{\sigma}\psi' - i\sqrt{2}(\bar{\phi}'\psi'\epsilon\lambda - \bar{\lambda}\epsilon\bar{\psi}'\phi') \quad (8.68)$$

remains the same. When we apply the gauge transformed fields, the Lagrangian doesn't change.

8.4.1. General Case with Interaction Terms

Now, we make a change of the fields ϕ and ψ in the following forms:

$$\phi = e^{-i\int^t A(t')dt'}\Phi \quad (8.69)$$

$$\psi = e^{-i\int^t A(t')dt'}\Psi \quad (8.70)$$

So we will not see the gauge potential in the Lagrangian since the covariant derivative now becomes usual derivative with respect to time:

$$D_t\phi = \dot{\Phi} \quad (8.71)$$

Lagrangian is rewritten

$$L = \frac{\mu}{2}(\dot{\mathbf{x}}^2 + D^2 + 2i\bar{\lambda}\lambda) - \theta D + \dot{\Phi}\dot{\bar{\Phi}} - (\mathbf{x}^2 + D)\Phi\bar{\Phi} + i\bar{\Psi}\dot{\Psi} - \bar{\Psi}x_i\sigma_i\Psi - i\sqrt{2}(\bar{\Phi}\Psi\epsilon\lambda - \bar{\lambda}\epsilon\bar{\Psi}\Phi) \quad (8.72)$$

where $\omega^2 = \mathbf{x}^2 + D$. We define $\alpha = \epsilon\bar{\lambda}$ and treat λ as a constant. Equations of motion are

$$\begin{aligned} \ddot{x}_i + \frac{2}{\mu} x_i \bar{\Phi}\Phi + \bar{\Psi}\hat{x}_i\sigma_i\Psi &= 0 & (8.73)\\ \ddot{\Phi} + \omega^2\Phi + i\sqrt{2}\bar{\alpha}\Psi &= 0 & (8.74)\\ \dot{\Psi} + ix_i\sigma_i\Psi + \sqrt{2}\alpha\Phi &= 0 & (8.75)\end{aligned}$$

Solutions of the equations can be expressed in the following form

$$\Psi(t) = \tilde{\psi}(t)e^{-ix_i\sigma_i t} \tag{8.76}$$

where

$$\tilde{\psi}(t) = \psi_0 - \sqrt{2}\int_0^t dt'\Phi(t')e^{ix_i\sigma_i t'}\alpha. \tag{8.77}$$

Explicitly,

$$\begin{aligned} \Psi(t) &= -\sqrt{2}(\cos xt - i\hat{x}^i\sigma_i\sin xt)\int_0^t dt'\Phi(t')(\cos xt' + i\hat{x}^j\sigma_j\sin xt')\alpha \\ &\quad +\psi_0 e^{-ix_i\sigma_i t}. & (8.78)\end{aligned}$$

We use the identity $\hat{x}^i\hat{x}^j\sigma_i\sigma_j = 1$ and get

$$\begin{aligned} &= -\sqrt{2}(\int_0^t dt'\cos x(t-t')\Phi(t') + i\hat{x}_i\sigma_i\int_0^t dt'\sin x(t-t')\Phi(t'))\alpha \\ &\quad +\psi_0 e^{-ix_i\sigma_i t}. & (8.79)\end{aligned}$$

Plug it in (8.74):

$$\begin{aligned} \ddot{\Phi} + w^2\Phi &= 2i\bar{\alpha}\int_0^t dt'\cos x(t-t')\Phi(t')\alpha - 2\bar{\alpha}\hat{x}_i\sigma_i\int_0^t dt'\sin x(t-t')\Phi(t')\alpha \\ &\quad -i\sqrt{2}\bar{\alpha}\psi_0\cos xt - \sqrt{2}\bar{\alpha}\hat{x}_i\sigma_i\psi_0\sin xt. & (8.80)\end{aligned}$$

From now on we define $\hat{\chi} = \hat{x}_i \sigma_i$. The right hand side of the equation involves fermionic terms so for a solution to this kind of equation we make a quasi-classical expansion of the Φ field :

$$\begin{aligned}\Phi(t) &= \phi_{\mathrm{qc}} + \varphi_0 \bar{\alpha}\alpha + \varphi_1 \bar{\alpha}\hat{\chi}\alpha + \varphi_4 (\bar{\alpha}\alpha)^2 + \gamma_0 \bar{\alpha}\psi_0 + \gamma_1 \bar{\alpha}\hat{\chi}\psi_0 + \gamma_4 (\bar{\alpha}\psi_0)^2 \\ &\quad + \eta_0 \bar{\alpha}\alpha\bar{\alpha}\psi_0 + \eta_1 \bar{\alpha}\alpha\bar{\alpha}\hat{\chi}\psi_0 \end{aligned} \tag{8.81}$$

Some terms are not in the expansion since they are zero or written in the form of other terms. Here are some identities to calculate the terms in the expansion:

$$\bar{\alpha}\hat{\chi}\alpha\bar{\alpha}\alpha = 0 \tag{8.82}$$

$$\bar{\alpha}\hat{\chi}\alpha\bar{\alpha}\hat{\chi}\alpha = -(\bar{\alpha}\alpha)^2 \tag{8.83}$$

$$\bar{\alpha}\hat{\chi}\alpha\bar{\alpha}\psi_0 = -\bar{\alpha}\alpha\bar{\alpha}\hat{\chi}\psi_0 \tag{8.84}$$

$$\bar{\alpha}\hat{\chi}\psi_0\bar{\alpha}\hat{\chi}\psi_0 = -(\bar{\alpha}\psi_0)^2 \tag{8.85}$$

$$\bar{\alpha}\hat{\chi}\alpha\bar{\alpha}\hat{\chi}\psi_0 = -\bar{\alpha}\alpha\bar{\alpha}\psi_0 \tag{8.86}$$

Each term in the expansion are independent. Plug it in (8.80) so we obtain 9 linearly independent equations:

$$\ddot{\phi}_{\mathrm{qc}} + \omega^2 \phi_{\mathrm{qc}} = 0 \tag{8.87}$$

$$\ddot{\varphi}_0 + \omega^2 \varphi_0 = 2i \int_0^t \cos x(t-t')\phi_{qc}(t')dt' \tag{8.88}$$

$$\ddot{\varphi}_1 + \omega^2 \varphi_1 = 2 \int_0^t \sin x(t-t')\phi_{qc}(t')dt' \tag{8.89}$$

$$\ddot{\varphi}_4 + \omega^2 \varphi_4 = 2i \int_0^t \cos x(t-t')\varphi_0(t')dt' - 2\int_0^t \sin x(t-t')\varphi_1(t')dt' \tag{8.90}$$

$$\ddot{\gamma}_0 + \omega^2 \gamma_0 = -i\sqrt{2}\cos xt \tag{8.91}$$

$$\ddot{\eta}_0 + \omega^2 \eta_0 = 2i \int_0^t \cos x(t-t')\gamma_0(t')dt' - 2\int_0^t \sin x(t-t')\gamma_1(t')dt' \tag{8.92}$$

$$\ddot{\eta}_1 + \omega^2 \eta_1 = 2i \int_0^t \cos x(t-t')\gamma_1(t')dt' - 2\int_0^t \sin x(t-t')\gamma_0(t')dt' \tag{8.93}$$

$$\ddot{\gamma}_4 + \omega^2 \gamma_4 = 0 \tag{8.94}$$

$$\ddot{\gamma}_1 + \omega^2 \gamma_1 = -\sqrt{2}\sin xt \tag{8.95}$$

The solutions are:

$$\begin{aligned}
\phi_{\rm qc}(t) &= \gamma_4(t) = A\cos w(t-t_0) + B\sin w(t-t_0) && (8.96)\\
\varphi_0(t) &= \frac{1}{2w(x^2-w^2)^2}(-4iAw^2\cos(t-t_0)x - 4ixBw\sin(t-t_0)x)\\
&\quad -\frac{i}{2w(x^2-w^2)}\sin w(t-t_0)(A+2w(tB+iB'(x^2-w^2))\\
&\quad -\frac{i}{2w(x^2-w^2)}\cos w(t-t_0)(A+2w(-tB+A'(x^2-w^2))) && (8.97)\\
\varphi_1(t) &= (x^2-w^2)(2A'x^2w^2 - 2A'xw^4(A+2wtB))\cos(t-t_0)w && (8.98)\\
&\quad +4w^2(Ax\cos(t-t_0)x - Bw\sin(t-t_0)x))\\
&\quad \frac{1}{2w^2x^2-2w^4}(2B'x^2w^2 - 2B'w^4 - x(A+2wtB))\sin(t-t_0)w\\
\gamma_1(t) &= A'\cos(t-t_0)w + B'\sin(t-t_0)w + \frac{2\sin(t-t_0)x}{x^2-w^2} && (8.99)\\
\gamma_0(t) &= A'\cos(t-t_0)w + B'\sin(t-t_0)w - 2i\frac{\cos(t-t_0)x}{x^2-w^2} && (8.100)
\end{aligned}$$

We will not write the other solutions, since they are really complicated and long. They have more time dependence because integration involves the solutions with the first order time terms. The average of Φ equation is:

$$\begin{aligned}
<\bar{\Phi}\Phi> &= <\bar{\phi}_{\rm qc}\phi_{qc}> + \bar{\alpha}\alpha(<\bar{\phi}_{qc}\varphi_0> + <\bar{\varphi}_0\phi_{qc}>)\\
&\quad +(\bar{\alpha}\alpha)^2(<\bar{\phi}_{qc}\varphi_4> + <\bar{\varphi}_4\phi_{qc}> - <\bar{\varphi}_1\varphi_1> + <\bar{\varphi}_0\varphi_0>)\\
&\quad +\bar{\alpha}\hat{\chi}\alpha(<\bar{\phi}_{qc}\varphi_1> + <\bar{\varphi}_1\phi_{qc}>)\\
&\quad +\bar{\alpha}\psi_0<\bar{\phi}_{qc}\gamma_0> + \bar{\psi}_0\alpha<\bar{\gamma}_0\phi_{qc}> + \bar{\psi}_0\hat{\chi}\alpha<\bar{\gamma}_1\phi_{qc}>\\
&\quad +\bar{\alpha}\hat{\chi}\psi_0<\bar{\phi}_{qc}\gamma_1> + (\bar{\alpha}\psi_0)^2<\bar{\phi}_{qc}\gamma_4> + (\bar{\psi}_0\alpha)^2<\bar{\gamma}_4\phi_{qc}>\\
&\quad +\bar{\alpha}\alpha\bar{\alpha}\psi_0(<\bar{\phi}_{qc}\eta_0> + <\bar{\varphi}_0\gamma_0> - <\bar{\varphi}_1\gamma_1>)\\
&\quad +\bar{\psi}_0\alpha\bar{\alpha}\alpha(<\bar{\gamma}_0\varphi_0> - <\bar{\gamma}_1\varphi_1> + <\bar{\eta}_0\phi_{qc}>)\\
&\quad +\bar{\psi}_0\alpha\bar{\alpha}\psi_0(<\bar{\gamma}_0\gamma_0> - <\bar{\gamma}_1\gamma_1>)\\
&\quad -\frac{1}{2}\bar{\psi}_0\psi_0(\bar{\alpha}\alpha)^2(<\bar{\eta}_0\gamma_0> + <\bar{\gamma}_0\eta_0> - <\bar{\eta}_i\gamma_1> - <\bar{\gamma}_1\eta_1>)\\
&\quad +(\bar{\psi}_0\alpha)^2\bar{\alpha}\psi_0<\bar{\gamma}_4\gamma_0> + \bar{\psi}_0\alpha(\bar{\alpha}\psi_0)^2<\bar{\gamma}_0\gamma_4> + (\bar{\psi}_0\alpha)^2(\bar{\alpha}\psi_0)^2<\bar{\gamma}_4\gamma_4>\\
&\quad +\bar{\alpha}\alpha\bar{\alpha}\hat{\chi}\psi_0(<\bar{\phi}_{qc}\eta_1> + <\bar{\varphi}_0\gamma_1> + <\bar{\varphi}_1\gamma_0>)\\
&\quad +(\bar{\psi}_0\alpha)^2\bar{\alpha}\hat{\chi}\psi_0<\bar{\gamma}_4\gamma_1> + \bar{\psi}_0\hat{\chi}\alpha\bar{\alpha}\alpha(<\bar{\eta}_1\phi_{qc}> + <\bar{\gamma}_1\varphi_0> - <\bar{\gamma}_0\varphi_1>)
\end{aligned}$$

$$
\begin{aligned}
&+\bar{\psi}_0\alpha\bar{\alpha}\hat{\chi}\psi_0(<\bar{\gamma}_0\gamma_1> + <\bar{\gamma}_1\gamma_0>) + \bar{\psi}_0\hat{\chi}\alpha(\bar{\alpha}\psi_0)^2 <\bar{\gamma}_1\gamma_4> \\
&-\frac{1}{2}(\bar{\alpha}\alpha)^2\bar{\psi}_0\hat{\chi}\psi_0(<\bar{\gamma}_0\eta_1> + <\bar{\eta}_0\gamma_1> + <\bar{\gamma}_1\eta_0>) \qquad (8.101)
\end{aligned}
$$

To find it we used the additional identities and some derivations in Appendix [3] :

$$
\begin{aligned}
\bar{\alpha}\alpha\bar{\alpha}\psi_0\bar{\psi}_0\alpha &= -\frac{1}{2}(\bar{\alpha}\alpha)^2\bar{\psi}_0\psi_0 && (8.102)\\
\bar{\alpha}\alpha\bar{\alpha}\psi_0\bar{\psi}_0\hat{\chi}\alpha &= \bar{\alpha}\alpha\bar{\psi}_0\alpha\bar{\alpha}\hat{\chi}\psi_0 = -\frac{1}{2}(\bar{\alpha}\alpha)^2\bar{\psi}_0\hat{\chi}\psi_0 && (8.103)\\
\bar{\alpha}\alpha\bar{\alpha}\hat{\chi}\psi_0\bar{\psi}_0\hat{\chi}\alpha &= -\frac{1}{2}(\bar{\alpha}\alpha)^2\bar{\psi}_0\psi_0 && (8.104)\\
(\bar{\alpha}\psi_0)^2\bar{\psi}_0\hat{\chi}\alpha &= -\bar{\alpha}\alpha\bar{\alpha}\psi_0\bar{\psi}_0\hat{\chi}\psi_0 && (8.105)\\
(\bar{\psi}_0\alpha)^2\bar{\alpha}\hat{\chi}\psi_0 &= -\bar{\alpha}\alpha\bar{\psi}_0\alpha\bar{\psi}_0\hat{\chi}\psi_0 && (8.106)
\end{aligned}
$$

The average for Ψ equation is :

$$
\begin{aligned}
<\bar{\Psi}\hat{\chi}\Psi> &= <e^{i\chi t}\bar{\tilde{\psi}}\hat{\chi}e^{-i\chi t}\tilde{\psi}> \\
&= <(\cos xt + i\hat{\chi}\sin xt)\bar{\tilde{\psi}}\hat{\chi}(\cos xt - i\hat{\chi}\sin xt)\tilde{\psi}> \\
&= <\bar{\tilde{\psi}}\hat{\chi}\tilde{\psi}> \qquad (8.107)
\end{aligned}
$$

Let us compute

$$
\begin{aligned}
\bar{\tilde{\psi}}\tilde{\psi} &= (\bar{\psi}_0 - \sqrt{2}\bar{\alpha}\int_0^t dt'\bar{\Phi}(t')e^{-ix_i\sigma_i t'})(\psi_0 - \sqrt{2}\int_0^t dt''\Phi(t'')e^{ix_i\sigma_i t''}\alpha) \\
&= \bar{\psi}_0\psi_0 + 2\int_0^t\int_0^t(\bar{\alpha}\alpha\cos x(t'-t'') + i\bar{\alpha}\hat{\chi}\alpha\sin x(t'-t'')\Phi(t')\bar{\Phi}(t''))dt'dt'' \\
&\quad -\sqrt{2}\int_0^t((\bar{\psi}_0\alpha\Phi(t') + \bar{\alpha}\psi_0\bar{\Phi}(t'))\cos xt' + i(\bar{\psi}_0\hat{\chi}\alpha\Phi(t') + \bar{\alpha}\hat{\chi}\psi_0\bar{\Phi}(t'))\sin xt')dt'
\end{aligned}
$$

For now, we stop our discussion here since it has some secular terms.

[3] see (E.5)-(E.9)

9. CONCLUSIONS

In this thesis SUSY transformations are applied on the Lagrangian with a Dirac monopole term and some conditions on the Lagrangian are determined to preserve the symmetry. For Quiver and Dirac monopole Lagrangian SUSY solutions are calculated. Under some QM approximations these two Lagrangians are similar to each other. To get the same result by classical way, we derive adiabatic invariant theorem for a fermionic and bosonic system and use them in separation of scales. Effective potentials derived from the toy models are consistent with the potentials of chiral modes in the Quiver effective Lagrangian. For the general Lagrangian with the interaction term between vector and chiral multiplets we separate the scales: $\mathbf{x}$ and λ are slowly changing fields and chiral multiplets are rapidly oscillating fields. We plan plugging the averages of chiral multiplet into (8.73) and (7.3)

$$\ddot{x}_i + \frac{2}{\mu} x_i < \bar{\Phi}\Phi > + < \bar{\Psi}\hat{\chi}\Psi > \; = \; 0 \tag{9.1}$$

$$\dot{\lambda} + \frac{\sqrt{2}}{\mu}\epsilon < \bar{\Psi}\Phi > \; = \; 0 \tag{9.2}$$

to find effective time independent potentials. The quasi-classical expansion of the complex scalar field Φ gives rise to nine linearly independent equations. However, some solutions involves problematic secular terms with the periodic oscillation between the (8.88)-(8.95). Since we should deal with the secular terms which are growing in time, we decide to stop here anymore and use our results for a future study to find a suitable solution.

APPENDIX A: NOTATIONS AND CONVENTIONS

Vectors are written in bold letters. Spinors are denoted by λ, fermions are denoted by ψ and $\bar{\psi}$. The unbarred spinors have indices down,the barred ones indices up. They are related through complex conjugation $(\psi_\alpha)^* = \bar{\psi}^\alpha$. The notations are

$$\bar{\psi}\chi = \bar{\psi}^\alpha \chi_\alpha = -\chi_\alpha \bar{\psi}^\alpha = -\chi\bar{\psi} \tag{A.1}$$

$$\bar{\psi}\sigma^i \chi = \bar{\psi}^\alpha (\sigma^i)_\alpha^\beta \chi_\beta \tag{A.2}$$

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, \qquad \epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha^\beta, \qquad \epsilon^{12} = 1, \quad \epsilon_{12} = -1 \tag{A.3}$$

$$(\epsilon\psi)^\alpha = \epsilon^{\alpha\beta}\psi_\beta \tag{A.4}$$

$$(\bar{\psi}\epsilon)_\alpha = \bar{\psi}^\beta \epsilon_{\beta\alpha} \tag{A.5}$$

$$\psi\epsilon\chi = \psi_\alpha \epsilon^{\alpha\beta}\chi_\beta \tag{A.6}$$

For Grassmann variables interchange of order comes with a minus sign.

$$\bar{\lambda}^\alpha \lambda_\alpha = -\lambda_\alpha \bar{\lambda}^\alpha \tag{A.7}$$

where $\alpha = 1, 2$. We define

$$\bar{\lambda}^\alpha = \epsilon^{\alpha\beta}\lambda_\beta, \quad \lambda_\alpha = \epsilon_{\alpha\beta}\bar{\lambda}^\beta \tag{A.8}$$

for the supersymmetry conservation section in Chapter 6. The convention here is that adjacent indices are always contracted putting the epsilon tensor on the left.

$$\bar{\lambda}^{\alpha}\bar{\lambda}^{\beta} = |\bar{\lambda}|^2\epsilon^{\alpha\beta} \tag{A.9}$$

Some identities:

$$\epsilon\lambda\psi_0 = \epsilon^{\alpha\beta}\lambda_\beta(\psi_0)_\alpha = -\lambda_\beta\epsilon^{\beta\alpha}(\psi_0)_\alpha = -\lambda\epsilon\psi_0 \tag{A.10}$$

$$\psi_0\epsilon\lambda = \lambda\epsilon\psi_0 = -\epsilon\lambda\psi_0 \quad \bar{\psi}_0\epsilon\bar{\lambda} = \bar{\lambda}\epsilon\bar{\psi}_0 \tag{A.11}$$

The σ_i Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.12}$$

$$\sigma_j^\dagger = \sigma_j \tag{A.13}$$

We assume Grassmann variables interchange under * with no additional minus:

$$\begin{aligned} \overline{\psi_\alpha\epsilon^{\alpha\beta}\lambda_\beta} &= \lambda^*_\beta(\epsilon^{\alpha\beta})^*\psi^*_\alpha = \bar{\lambda}^\beta\epsilon_{\beta\alpha}\bar{\psi}^\alpha && \text{(A.14)} \\ (\epsilon^{\alpha\beta})^* &= -\epsilon_{\alpha\beta} = \epsilon_{\beta\alpha} && \text{(A.15)} \\ (\sigma^\beta_\alpha)^* &= \sigma^\alpha_\beta && \text{(A.16)} \end{aligned}$$

APPENDIX B: LAGRANGIAN OF A PARTICLE IN A BACKGROUND MAGNETIC FIELD

The Lagrangian for a charge q moving in a magnetic field is

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + \frac{q}{c}\mathbf{A}.\dot{\mathbf{x}} - q\phi \tag{B.1}$$

where $\mathbf{A}$ is a vector potential and ϕ is a scalar potential. Hamiltonian of the system is defined as

$$H(\mathbf{p},\mathbf{x}) = \mathbf{p}.\dot{\mathbf{x}} - L \tag{B.2}$$

with canonical momentum $\mathbf{p} = m\dot{\mathbf{x}} + \frac{q}{c}\mathbf{A}$. We replace velocities with momenta and obtain the Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - \frac{q}{c}\mathbf{A})^2 + q\phi. \tag{B.3}$$

We must now check that Hamilton equations reproduce the Lorentz force law. The first equation is

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i - \frac{q}{c}A_i}{m}. \tag{B.4}$$

The second Hamilton equation is

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = \frac{q}{mc}(p_j - \frac{q}{c}A_j)\frac{\partial A_j}{\partial x_i} - q\frac{\partial \phi}{\partial x_i}. \tag{B.5}$$

Now plug (10.4) in (10.5) and get

$$\dot{p}_i = \frac{q}{c}\dot{x}_j\frac{\partial A_j}{\partial x_i} - q\frac{\partial \phi}{\partial x_i}. \tag{B.6}$$

Now differentiate the first Hamilton equation with respect to time

$$m\ddot{x}_i = \dot{p}_i - \frac{q}{c}\frac{\partial A_i}{\partial t} - \frac{q}{c}\frac{\partial A_i}{\partial x_j}\dot{x}_j \tag{B.7}$$

and obtain:

$$m\ddot{x}_i = \frac{q}{c}\dot{x}_j(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}) + qE_i \tag{B.8}$$

where $E_i = -\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{\partial \phi}{\partial x_i}$. Note that

$$\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = (\delta_{ir}\delta_{jq} - \delta_{iq}\delta_{jr})\frac{\partial A_q}{\partial x_r} = \epsilon_{ijk}\epsilon_{krq}\frac{\partial A_q}{\partial x_r} \tag{B.9}$$

we find

$$m\ddot{x}_i = \frac{q}{c}\epsilon_{ijk}\dot{x}_j\epsilon_{krq}\frac{\partial A_q}{\partial x_r} + qE_i \tag{B.10}$$

Now, we use the identity $(\nabla \times \mathbf{A})_i = \epsilon_{ijk}\partial_j A_k$ to obtain the Lorentz force

$$m\ddot{x}_i = \frac{q}{c}(\dot{\mathbf{x}} \times (\nabla \times \mathbf{A}))_i + qE_i \tag{B.11}$$

where the magnetic field is $\mathbf{B} = \nabla \times \mathbf{A}$. More generally it yields

$$\mathbf{F} = \frac{q}{c}(\dot{\mathbf{x}} \times \mathbf{B}) + q\mathbf{E}. \tag{B.12}$$

B.1. Angular Momentum of An Electric and A Magnetic charge

Dirac put Maxwell equations in a symmetric form by adding magnetic monopoles to the theory. Consider a monopole p sitting at r=0. Similar to the Gauss law for electric charge, divergence of magnetic field gives us magnetic charge.

$$\boldsymbol{\nabla}.\mathbf{B} = 4\pi\frac{g}{c}\delta^3(r) \tag{B.13}$$

using the identities $\nabla^2(\frac{1}{r}) = -4\pi\delta^3(r)$ and $\boldsymbol{\nabla}(\frac{1}{r}) = -\frac{\mathbf{r}}{r^3}$ we find

$$\mathbf{B} = \frac{g}{c}\frac{\mathbf{r}}{r^3}. \tag{B.14}$$

Suppose we have an electric charge q and a magnetic monopole g which are separated by a distance d along z axis. The field of the electric charge is

$$\mathbf{E} = q\frac{\mathbf{r}}{r^3} \tag{B.15}$$

and the field of the magnetic charge is

$$\mathbf{B} = \frac{g}{c}\frac{\mathbf{r}^{'}}{r'^3} \tag{B.16}$$

$$\mathbf{B} = \frac{g}{c}\frac{\mathbf{r} - d\hat{z}}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}}. \tag{B.17}$$

Momentum density is

$$\overrightarrow{\mathbb{P}} = \epsilon_0(\mathbf{E} \times \mathbf{B}) = \frac{\epsilon_0}{c}qg\frac{(-d)(\mathbf{r} \times \hat{z})}{r^3(r^2 + d^2 - 2rd\cos\theta)^{3/2}}. \tag{B.18}$$

Angular momentum density:

$$\overrightarrow{\ell} = \mathbf{r} \times \overrightarrow{\mathbb{P}} = -\frac{\epsilon_0}{c}qgd\frac{\mathbf{r} \times (\mathbf{r} \times \hat{z})}{r^3(r^2 + d^2 - 2rd\cos\theta)^{3/2}}. \tag{B.19}$$

We use the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}.\mathbf{C}) - \mathbf{C}(\mathbf{A}.\mathbf{B})$ so we have

$$\mathbf{r} \times (\mathbf{r} \times \hat{z}) = \mathbf{r}(\mathbf{r}.\hat{z}) - \mathbf{r}^2\hat{z} = r^2\cos\theta\hat{r} - \mathbf{r}^2\hat{z}. \tag{B.20}$$

The x and y components will integrate to zero, using $(\hat{r}_z) = \cos\theta$. We have

$$\mathbf{L} = -\frac{\epsilon_0}{c} epd\hat{z} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{r^2(\cos^2\theta - 1)}{r^3(r^2 + d^2 - 2rd\cos\theta)^{3/2}} r^2 \sin\theta dr d\theta d\phi$$

change the variable $u = \cos\theta$:

$$\mathbf{L} = \frac{\epsilon_0}{c} qgd\hat{z} 2\pi \int_{-1}^{+1} \int_0^{\infty} \frac{r(1-u^2)}{(r^2 + d^2 - 2rdu)^{3/2}} dudr$$

$$\int_0^{\infty} \frac{r}{(r^2 + d^2 - 2rdu)^{3/2}} dr = \frac{ru - d}{d(1-u^2)\sqrt{r^2 + d^2 - 2rdu}} \Big|_0^{\infty}$$

$$= \frac{u}{d(1-u^2)} + \frac{d}{d(1-u^2)d} = \frac{u+1}{d(1-u^2)} = \frac{1}{d(1-u)}$$

$$\mathbf{L} = \frac{2\pi\epsilon_0}{c} qgd\hat{z} \frac{1}{d} \int_{-1}^{+1} \frac{1-u^2}{1-u} du = \frac{2\pi\epsilon_0}{c} qg\hat{z} \frac{1}{d} \int_{-1}^{+1} (1+u) du$$

$$= \frac{2\pi\epsilon_0}{c} qg\hat{z}(u + \frac{u^2}{2})|_{-1}^{+1} = \frac{1}{c} qg\hat{z} \tag{B.21}$$

where $\epsilon_0 = 1/4\pi$.

B.2. Quantization Condition for Magnetic Monopoles

Quantum mechanically angular momentum is quantized.

$$L = \frac{n\hbar}{2} \Rightarrow 2L \in \mathbb{Z} \tag{B.22}$$

This is equivalent to $\dfrac{2qg}{c} \in \mathbb{Z}$. If there exists some monopoles somewhere in the universe all electric charges are quantized and the magnetic charge takes discrete values

$$g = \frac{\hbar c n}{2q}, \qquad n \in \mathbb{Z}. \tag{B.23}$$

B.3. Gauge Potential and Charge Quantization

The relation between the magnetic field and the vector potential is given by $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. We define two vector potential on the sphere which is valid for both sides of the equator. $\mathbf{A}^N$ is the north and $\mathbf{A}^S$ is the south part of the sphere. We introduce A^N in terms of components by [5]

$$A^N_x = -\frac{gy}{r(r+z)}, \qquad A^N_y = \frac{gx}{r(r+z)}, \qquad A^N_z = 0 \tag{B.24}$$

Using the identity we find magnetic field for the north part

$$\boldsymbol{\nabla} \times \mathbf{A}^N = -\hat{x}\partial_z(\frac{gx}{r(r+z)}) - \hat{y}\partial_z(\frac{gy}{r(r+z)}) + \hat{z}[\partial_x(\frac{gx}{r(r+z)}) + \partial_y(\frac{gy}{r(r+z)})] \tag{B.25}$$

with the help of following derivations and using $\partial_z r = \frac{z}{r}$, $\partial_x r = \frac{x}{r}$, $\partial_y r = \frac{y}{r}$

$$\begin{aligned}
\partial_z(\frac{gx}{r(r+z)}) &= -\frac{gx}{(r^2+rz)^2}(2r\partial_z r + \partial_z r z + r), \\
\partial_z(\frac{gy}{r(r+z)}) &= -\frac{gy}{(r^2+rz)^2}(2r\partial_z r + \partial_z r z + r) \\
\partial_x(\frac{gx}{r(r+z)}) &= \frac{g}{r^2+rz} - \frac{gx}{(r^2+rz)^2}(2r\partial_x r + \partial_x r z), \\
\partial_y(\frac{gy}{r(r+z)}) &= \frac{g}{r^2+rz} - \frac{gy}{(r^2+rz)^2}(2r\partial_y r + \partial_y r z)
\end{aligned} \tag{B.26}$$

we obtain

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{A}^N &= \frac{g}{(r^2+rz)^2}[(2r\frac{z}{r} + \frac{z^2}{r} + r)(x\hat{x} + y\hat{y}) + \hat{z}(2r^2 + 2rz - (x^2+y^2)(2 + \frac{z}{r})) \\
&= \frac{g}{(r^2+rz)^2}[(\frac{(r+z)^2}{r})(x\hat{x} + y\hat{y}) + \hat{z}(2r^2 + 2rz - (r^2 - z^2)(2 + \frac{z}{r}))]
\end{aligned}$$

Finally we obtain magnetic field

$$\nabla \times \mathbf{A}^N = g\frac{\mathbf{r}}{r^3} + 4\pi g\delta(x)\delta(y)\theta(-z) \tag{B.27}$$

except along the negative z axis ($\theta = \pi$). The singularity along the z axis is called the Dirac string. Instead if we choose the south of the hemisphere vector potential is

$$A_x^S = -\frac{gy}{r(r-z)}, \qquad A_y^S = -\frac{gx}{r(r-z)}, \qquad A_z^S = 0 \tag{B.28}$$

This time magnetic field is not defined along the positive z axis ($\theta = 0$). The reason for singularity is

$$\phi = \oint_S \mathbf{B}.d\mathbf{S} = \int_S (\nabla \times \mathbf{A}).dS = \int_V \nabla.(\nabla \times \mathbf{A})dV = 0 \tag{B.29}$$

In polar coordinates

$$A_x^N = -\frac{g\sin\theta\sin\phi}{r(1+\cos\theta)} \tag{B.30}$$

$$A_y^N = \frac{g\sin\theta\cos\phi}{r(1+\cos\theta)} \tag{B.31}$$

$$A_z^N = 0 \tag{B.32}$$

and similarly,

$$A_x^S = \frac{g\sin\theta\sin\phi}{r(1-\cos\theta)} \tag{B.33}$$

$$A_y^S = -\frac{g\sin\theta\cos\phi}{r(1-\cos\theta)} \tag{B.34}$$

$$A_z^S = 0. \tag{B.35}$$

use the polar coordinate $\hat{e}_\phi = -\sin\phi\hat{x} + \cos\phi\hat{y}$

$$\begin{aligned}\mathbf{A}^N &= \frac{g\sin\theta}{r(1+\cos\theta)}(-\sin\phi\hat{x} + \cos\phi\hat{y}) = \frac{g\sin\theta}{r(1+\cos\theta)}\hat{e_\phi} = \frac{g\sin\theta(1-\cos\theta)}{r(1-\cos^2\theta)} \\ &= \frac{g(1-\cos\theta)}{r\sin\theta}\hat{e}_\phi \end{aligned} \tag{B.36}$$

$$\begin{aligned}\mathbf{A}^S &= -\frac{g\sin\theta}{r(1-\cos\theta)}(-\sin\phi\hat{x}+\cos\phi\hat{y}) = -\frac{g\sin\theta}{r(1-\cos\theta)}\hat{e_\phi} = -\frac{g\sin\theta(1+\cos\theta)}{r(1-\cos^2\theta)} \\ &= -\frac{g(1+\cos\theta)}{r\sin\theta}\hat{e}_\phi \end{aligned} \tag{B.37}$$

B.3.1. The Wu-Yang Monopole

Their aim was to define magnetic field with not only one vector field but two vector fields $\mathbf{A}^N$ and $\mathbf{A}^S$. The equator of the sphere is the boundary between $\mathbf{A}^N$ and $\mathbf{A}^S$. At overlap regions they give the same magnetic field. They are related to each other by a gauge transformation on the equator $\theta = \pi/2$ which is

$$\mathbf{A}^N - \mathbf{A}^S = \boldsymbol{\nabla} f$$

$$\mathbf{A}^N - \mathbf{A}^S = (\frac{g(1-\cos\theta}{r\sin\theta} + \frac{g(1+\cos\theta)}{r\sin\theta})\hat{e}_\phi = \frac{2p}{r\sin\theta}\hat{e}_\phi = 2g\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\phi\hat{e}_\phi \tag{B.38}$$

where we used $\boldsymbol{\nabla} = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{e}_\theta + \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\hat{e}_\phi$

$$\mathbf{A}^N - \mathbf{A}^S = 2g\boldsymbol{\nabla}\phi \tag{B.39}$$

when we match the two equation, we find $\boldsymbol{\nabla} f = \boldsymbol{\nabla}(2g\phi)$ which gives us gauge transformation factor $f = 2g\phi$. It is ill defined at $\theta = 0$ and $\theta = \pi$. Our Gauge transformation works at only at $\theta = \frac{\pi}{2}$. Total magnetic flux is now the sum of two contributions coming from the north and the south parts.

$$\phi = \oint(\boldsymbol{\nabla}\times\mathbf{A}).d\mathbf{S} = \int_N(\boldsymbol{\nabla}\times\mathbf{A}^N).d\mathbf{S} + \int_S(\boldsymbol{\nabla}\times\mathbf{A}^S).d\mathbf{S} \tag{B.40}$$

Stokes theorem yields

$$\phi = \oint_{equ}\mathbf{A}^N.d\boldsymbol{\ell} - \oint_{equator}\mathbf{A}^S.d\boldsymbol{\ell} = \oint_{equator}(\mathbf{A}^N - \mathbf{A}^S).d\boldsymbol{\ell} \tag{B.41}$$

$$\phi = \oint_{equator} \boldsymbol{\nabla}(2g\phi).d\boldsymbol{\ell} = 4\pi g, \qquad (\theta = \frac{\pi}{2}) \tag{B.42}$$

as we found before.

B.3.2. Charge Quantization From Schrodinger Equations

For a point particle with electric charge q and mass m moving in the field of a magnetic monopole of charge p Schrodinger eqn is

$$\frac{1}{2m}(\mathbf{p} - \frac{q}{c}\mathbf{A})^2\psi(r) = E\psi(r) \tag{B.43}$$

When we apply the gauge transformation on vector potential and wave function

$$\mathbf{A} \longrightarrow \mathbf{A} + \nabla f, \qquad |\psi\rangle \longrightarrow e^{i\beta}|\psi\rangle \tag{B.44}$$

, we will see Hamiltonian is invariant. We denote the transformed wave function as $\psi'(r) = e^{i\beta}\psi(r)$ where β is a phase factor. There is no change in $\langle\psi'|H'|\psi'\rangle = \langle\psi|H|\psi\rangle$

$$\langle\psi|e^{-i\beta}H'e^{i\beta}|\psi\rangle = \langle\psi|H|\psi\rangle \tag{B.45}$$

We see H' and H are related to each other in that $e^{-i\beta}H'e^{i\beta} = H$. Now gauge transformed Hamiltonian is denoted by $M'^2 = (\mathbf{p} - \frac{q}{c}\mathbf{A}')^2$. We rewrite $e^{-i\beta}M'^2e^{i\beta} = M^2$ and it yields

$$e^{-i\beta}M'e^{i\beta} = M \tag{B.46}$$

So we can write

$$e^{-i\beta}(\mathbf{p} - \frac{q}{c}\mathbf{A}')e^{i\beta}|\psi\rangle = (\mathbf{p} - \frac{q}{c}\mathbf{A})|\psi\rangle \tag{B.47}$$

$$(\mathbf{p} - \frac{q}{c}\mathbf{A}')e^{i\beta}|\psi\rangle = e^{i\beta}(\mathbf{p} - \frac{q}{c}\mathbf{A})|\psi\rangle \tag{B.48}$$

Momentum operator is $\mathbf{p} = -i\hbar\nabla$ plugging it in the above equation

$$\hbar\boldsymbol{\nabla}\beta e^{i\beta}|\psi\rangle + e^{i\beta}\mathbf{p}|\psi\rangle - \frac{q}{c}\mathbf{A}e^{i\beta}|\psi\rangle - \frac{q}{c}\boldsymbol{\nabla}f e^{i\beta}|\psi\rangle = e^{i\beta}\mathbf{p}|\psi\rangle - \frac{q}{c}e^{i\beta}\mathbf{A}|\psi\rangle \tag{B.49}$$

After cancellation of some terms

$$\hbar\boldsymbol{\nabla}\beta e^{i\beta}|\psi\rangle - \frac{q}{c}\boldsymbol{\nabla}f e^{i\beta}|\psi\rangle = 0 \tag{B.50}$$

$$\hbar\boldsymbol{\nabla}\beta = \frac{q}{c}\boldsymbol{\nabla}f \tag{B.51}$$

we find the phase factor $\beta = qf/\hbar c$ and rewrite it in the transformed wave function $|\psi\rangle \longrightarrow e^{i\frac{ef}{\hbar c}}|\psi\rangle$ and we already found gauge transformation factor f putting them all together

$$|\psi^S\rangle = e^{\frac{-2iqg}{\hbar c}\phi}|\psi^N\rangle \tag{B.52}$$

where $|\psi^S\rangle$ and $|\psi^N\rangle$ are the wave functions of the southern and northern hemisphere. If we go around the equator from $\phi \to \phi + 2\pi$

$$|\psi^S\rangle = e^{(\frac{-2iqg}{\hbar c}\phi - \frac{4\pi iqg}{\hbar c})}|\psi^N\rangle \tag{B.53}$$

wave function must be single valued $|\psi^S(\phi+2\pi)\rangle = |\psi^S(\phi)\rangle$ so it requires the condition

$$\frac{4\pi qg}{\hbar c} = 2\pi n \tag{B.54}$$

which is $qg = \frac{n\hbar c}{2}$ Again we see g and q are quantized.

APPENDIX C: FERMIONIC TRANSFORMATION

The relation between the fermionic transformation and time translation is given as

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]f = \delta_s f \tag{C.1}$$

$$(\delta_{\epsilon_1}\delta_{\epsilon_2} f - \delta_{\epsilon_2}\delta_{\epsilon_1} f) = \delta_s f \tag{C.2}$$

$$-\{\epsilon_1 Q + \bar{\epsilon_1}\overline{Q}, \{\epsilon_2 Q + \bar{\epsilon_2}\overline{Q}, f\}\} + \{\epsilon_2 Q + \bar{\epsilon_2}\overline{Q}, \{\epsilon_1 Q + \bar{\epsilon_1}\overline{Q}, f\}\} = -s\{H, f\}. \tag{C.3}$$

It is multiplied with a minus sign and the right hand side is obtained by using the fermionic transformation on a function f,

$$\begin{aligned} &= \{\epsilon_1 Q, \{\epsilon_2 Q + \bar{\epsilon_2}\overline{Q}, f\}\} + \{\bar{\epsilon_1}\overline{Q}, \{\epsilon_2 Q + \bar{\epsilon_2}\overline{Q}, f\}\} - \{\epsilon_2 Q, \{\epsilon_1 Q + \bar{\epsilon_1}\overline{Q}, f\}\} \\ &\quad -\{\bar{\epsilon_2}\overline{Q}, \{\epsilon_1 Q + \bar{\epsilon_1}\overline{Q}, f\}\} \end{aligned} \tag{C.4}$$

$$\begin{aligned} &= \epsilon_1\epsilon_2\{Q, \{Q, f\}\} + \epsilon_1\bar{\epsilon_2}\{Q, \{\overline{Q}, f\}\} + \bar{\epsilon_1}\epsilon_2\{\overline{Q}, \{Q, f\}\} + \bar{\epsilon_1}\bar{\epsilon_2}\{\overline{Q}, \{\overline{Q}, f\}\} \\ &\quad -\epsilon_2\epsilon_1\{Q, \{Q, f\}\} - \epsilon_2\bar{\epsilon_1}\{Q, \{\overline{Q}, f\}\} - \bar{\epsilon_2}\epsilon_1\{\overline{Q}, \{Q, f\}\} - \bar{\epsilon_2}\bar{\epsilon_1}\{\overline{Q}, \{\overline{Q}, f\}\} \end{aligned}$$

Using the anti-commutation of Grassmann variables ϵ_1 and ϵ_2 we rearrange it as fallows

$$\begin{aligned} &= \epsilon_1\epsilon_2\left(\{Q, \{Q, f\}\} + \{Q, \{Q, f\}\}\right) + \epsilon_1\bar{\epsilon_2}\left(\{Q, \{\overline{Q}, f\}\} + \{\overline{Q}, \{Q, f\}\}\right) \\ &\quad +\bar{\epsilon_1}\bar{\epsilon_2}\left(\{\overline{Q}, \{\overline{Q}, f\}\} + \{\overline{Q}, \{\overline{Q}, f\}\}\right) + \bar{\epsilon_1}\epsilon_2\left(\{\overline{Q}, \{Q, f\}\} + \{Q, \{\overline{Q}, f\}\}\right) \end{aligned} \tag{C.5}$$

TheJacobi identity is

$$(-1)^{|f|}\{Q,\{\overline{Q},f\}\}-\{\overline{Q},\{f,Q\}\}+(-1)^{|f|}\{f,\{Q,\overline{Q}\}\}=0 \tag{C.6}$$

or

$$(-1)^{|f|}\{Q,\{\overline{Q},f\}\}+\{\overline{Q},\{Q,f\}\}=(-1)^{|f|}\{\{Q,\overline{Q}\},f\} \tag{C.7}$$

where $\{\{Q,\overline{Q}\},f\}=-\{f,\{Q,\overline{Q}\}\}$ since $|\{Q,\overline{Q}\}|$ is zero.

$$\{\{Q,\overline{Q}\},f\}=\{Q,\{\overline{Q},f\}\}+(-1)^{|f|}\{\overline{Q},\{Q,f\}\}==-2i\{H,f\} \tag{C.8}$$

$$\{H,f\}=\frac{i}{2}(\{Q,\{\overline{Q},f\}\}+(-1)^{|f|}\{\overline{Q},\{Q,f\}\}) \tag{C.9}$$

Consider the case that f is a bosonic function, the first and third terms are zero.

$$\epsilon_1\epsilon_2\left(\{Q,\{Q,f\}\}\right)+\bar{\epsilon_1}\bar{\epsilon_2}\{\overline{Q},\{\overline{Q},f\}\}=0 \tag{C.10}$$

and finally we have

$$(\epsilon_1\bar{\epsilon_2}+\bar{\epsilon_1}\epsilon_2)\left(\{Q,\{\overline{Q},f\}\}+\{\overline{Q},\{Q,f\}\}\right)=-2i(\epsilon_1\bar{\epsilon_2}+\bar{\epsilon_1}\epsilon_2)\{H,f\} \tag{C.11}$$

where $s=2i(\epsilon_1\bar{\epsilon_2}-\epsilon_2\bar{\epsilon_1})$.

APPENDIX D: CALCULATIONS FOR SECTON 8.3

The calculations are

$$\mu\ddot{x}_i + \bar{\psi}_0 e^{i\mathbf{x}.\boldsymbol{\sigma}t}\sigma_i e^{-i\mathbf{x}.\boldsymbol{\sigma}t}\psi_0 = 0 \tag{D.1}$$

$$\mu\ddot{x}_i + \bar{\psi}_0(\cos xt + i\hat{x}_j\sigma_j \sin xt)\sigma_i(\cos xt - i\hat{x}_k\sigma_k \sin xt)\psi_0 = 0 \tag{D.2}$$

$$\begin{aligned}
\mu\ddot{x}_i + \bar{\psi}_0(\cos xt + i\hat{x}_j\sigma_j \sin xt)\sigma_i(\cos xt - i\hat{x}_k\sigma_k \sin xt)\psi_0 &= 0 \\
\mu\ddot{x}_i + \bar{\psi}_0(\cos^2(xt)\sigma_i - i\cos xt \sin xt\hat{x}_j(\sigma_i\sigma_j - \sigma_j\sigma_i) + \sin^2(xt)\hat{x}_j\hat{x}_k\sigma_j\sigma_i\sigma_k)\psi_0 &= 0 \\
\mu\ddot{x}_i + \bar{\psi}_0(\cos^2(xt)\sigma_i + \sin 2xt\hat{x}_j\epsilon_{ijk}\sigma_k + \sin^2(xt)\hat{x}_j\hat{x}_k(\delta_{ij} + i\epsilon_{jil}\sigma_l)\sigma_k)\psi_0 &= 0
\end{aligned}$$

$$\mu\ddot{x}_i + \bar{\psi}_0(\cos^2(xt)\sigma_i + \sin 2xt\hat{x}_j\epsilon_{ijk}\sigma_k + \sin^2(xt)\hat{x}_i\hat{x}_k\sigma_k + i\sin^2(xt)\hat{x}_j\hat{x}_k\epsilon_{jik}$$

$$-\sin^2(xt)\hat{x}_j\hat{x}_k\epsilon_{jil}\epsilon_{lkm}\sigma_m)\psi_0 = 0$$

The fifth term drops in the equation. We use the identity $\epsilon_{jil}\epsilon_{lkm} = \delta_{jk}\delta_{im} - \delta_{jm}\delta_{ik}$.

$$\mu\ddot{x}_i + \bar{\psi}_0(\cos^2(xt)\sigma_i + \sin 2xt\hat{x}_j\epsilon_{ijk}\sigma_k + \sin^2(xt)\hat{x}_i\hat{x}_k\sigma_k - \sin^2(xt)(\hat{x}_j\hat{x}_j\sigma_i - \hat{x}_m\hat{x}_i\sigma_m))\psi_0 = 0$$

and we get

$$\mu\ddot{x}_i + \bar{\psi}_0(\cos(2xt)\sigma_i + \sin(2xt)\epsilon_{ijk}\hat{x}_j\sigma_k + 2\sin^2(xt)\hat{x}_i\hat{x}_k\sigma_k)\psi_0 = 0. \tag{D.3}$$

APPENDIX E: IDENTITIES FOR SECTION 8.4

We derived some matrix identities in the last section to compute Φ and ψ

$$\hat{\chi}\hat{\chi} = \hat{x}^i\hat{x}^j\sigma_i\sigma_j = \mathbf{1} \tag{E.1}$$

$$i\sigma_2 = \epsilon \tag{E.2}$$

$$(\sigma_i)^T\epsilon = -\epsilon(\sigma_i) \tag{E.3}$$

$$Tr(\sigma_i\sigma_j) = 2\delta_{ij} \tag{E.4}$$

$$\begin{aligned}
\bar{\alpha}\hat{\chi}\alpha\bar{\alpha}\alpha &= \bar{\alpha}^p\hat{\chi}_{pq}\alpha_q\bar{\alpha}^r\alpha_r \\
&= -\epsilon_{pr}\bar{\alpha}^1\bar{\alpha}^2\epsilon_{qr}\alpha_1\alpha_2\hat{\chi}_{pq} \\
&= -(\hat{\chi}^T\epsilon)_{qr}\epsilon^T_{rq} \\
&= -Tr(\hat{\chi}^T) = 0
\end{aligned} \tag{E.5}$$

$$\begin{aligned}
\bar{\alpha}\hat{\chi}\alpha\bar{\alpha}\hat{\chi}\alpha &= \bar{\alpha}^m\hat{\chi}_{mn}\alpha_n\bar{\alpha}^q\hat{\chi}_{qr}\alpha_r \\
&= -\epsilon_{mq}\bar{\alpha}^1\bar{\alpha}^2\epsilon_{nr}\alpha_1\alpha_2\hat{\chi}_{mn}\hat{\chi}_{qr} \\
&= -\bar{\alpha}^1\bar{\alpha}^2\alpha_1\alpha_2Tr(\epsilon^T\hat{\chi}\epsilon\hat{\chi}^T) \\
&= -(\bar{\alpha}\alpha)^2
\end{aligned} \tag{E.6}$$

$$\bar{\alpha}\hat{\chi}\alpha\bar{\alpha}\psi = -\bar{\alpha}\alpha\bar{\alpha}\hat{\chi}\psi \tag{E.7}$$

One can generalize it to other cases:

$$\begin{aligned}\bar{\alpha}\hat{\chi}\psi\bar{\alpha}\hat{\chi}\psi &= -(\bar{\alpha}\psi)^2 && \text{(E.8)}\\ \bar{\alpha}\hat{\chi}\alpha\bar{\alpha}\hat{\chi}\psi &= -\bar{\alpha}\alpha\bar{\alpha}\psi && \text{(E.9)}\end{aligned}$$

Clearly, some terms are zero due to the anti-commutation of 2 component fermions:

$$\bar{\alpha}\hat{\chi}\alpha(\bar{\alpha}\alpha)^2 = \bar{\alpha}\hat{\chi}\alpha(\bar{\alpha}\psi_0)^2 = \bar{\alpha}\hat{\chi}\alpha(\bar{\alpha}\alpha\bar{\alpha}\psi_0) = \bar{\alpha}\hat{\chi}\alpha(\bar{\alpha}\alpha\bar{\alpha}\hat{\chi}\psi_0) = 0 \qquad \text{(E.10)}$$

Plug them in (8.76) to get Ψ equation

$$\begin{aligned}\Psi &= \sqrt{2}((\int_0^t \phi_{qc}\cos x(t-t')dt')\alpha - i(\int_0^t \phi_{qc}\sin x(t-t')dt'\hat{\chi}\alpha\\ &+(\int_0^t \varphi_0\cos x(t-t')dt' - i\int_0^t \varphi_1\sin x(t-t')dt')\bar{\alpha}\alpha\alpha\\ &+(i\int_0^t \varphi_0\sin x(t-t')dt' + \int_0^t \varphi_1\cos x(t-t')dt')\bar{\alpha}\hat{\chi}\alpha\alpha\\ &+(\int_0^t \gamma_0\cos x(t-t')dt' - i\int_0^t \gamma_1\sin x(t-t')dt')\bar{\alpha}\psi_0\alpha\\ &+(i\int_0^t \gamma_0\sin x(t-t')dt' + \int_0^t \gamma_1\cos x(t-t')dt')\bar{\alpha}\psi_0\hat{\chi}\alpha\\ &+\int_0^t \gamma_4\cos x(t-t')dt'(\bar{\alpha}\psi_0)^2\alpha\\ &-(i\int_0^t \eta_1\sin x(t-t')dt' + \int_0^t \eta_0\cos x(t-t')dt'\\ &+i\int_0^t \eta_0\sin x(t-t')dt')(\bar{\alpha}\alpha)^2\psi_0 - (\int_0^t \eta_1\cos x(t-t')dt')(\bar{\alpha}\alpha)^2\hat{\chi}\psi_0\\ &+i\int_0^t \eta_4\sin x(t-t')dt'(\bar{\alpha}\psi_0)^2\hat{\chi}\alpha) + \psi_0(\cos xt - i\hat{\chi}\sin xt) \qquad \text{(E.11)}\end{aligned}$$

The complex conjugate of Ψ :

$$\begin{aligned}\bar{\Psi} &= \sqrt{2}(\bar{\alpha}(\int \bar{\phi}_{qc}\cos x(t-t')dt') - i\bar{\alpha}\hat{\chi}(\int \bar{\phi}_{qc}\sin x(t-t')dt')\\ &+\bar{\alpha}\bar{\alpha}\alpha(\int \bar{\varphi}_0\cos x(t-t')dt' + i\int \bar{\varphi}_i\sin x(t-t')dt')\end{aligned}$$

$$+\bar{\alpha}\bar{\alpha}\hat{\chi}\alpha(-i\int\bar{\varphi}_0\sin x(t-t')dt'+\int\bar{\varphi}_i\cos x(t-t')dt')$$
$$+\bar{\alpha}\bar{\psi}_0\alpha(\int\bar{\gamma}_0\cos x(t-t')dt'+i\int\bar{\gamma}_i\sin x(t-t')dt')$$
$$+\bar{\alpha}\hat{\chi}\bar{\psi}_0\alpha(-i\int\bar{\gamma}_0\sin x(t-t')dt'+\int\bar{\gamma}_i\cos x(t-t')dt')$$
$$+\bar{\alpha}(\bar{\psi}_0\alpha)^2\int\bar{\gamma}_4\cos x(t-t')dt'-\bar{\psi}_0(\bar{\alpha}\alpha)^2(-i\int\bar{\eta}_i\sin x(t-t')dt'$$
$$+\int\bar{\eta}_0\cos x(t-t')dt'-i\int\bar{\eta}_0\sin x(t-t')dt')$$
$$-\bar{\psi}_0\hat{\chi}(\bar{\alpha}\alpha)^2(\int\bar{\eta}_i\cos x(t-t')dt')$$
$$-i\bar{\alpha}\hat{\chi}(\bar{\psi}_0\alpha)^2\int\bar{\eta}_4\sin x(t-t')dt')+\bar{\psi}_0(\cos xt+i\hat{\chi}\sin xt) \tag{E.12}$$

Collection of the similar matrix terms in averages for the last section of Chapter 8 :

$$\begin{aligned} A &= \hat{\chi}\bar{\alpha}\alpha+\hat{\chi}\bar{\psi}_0\alpha+\hat{\chi}\bar{\alpha}\psi_0+\bar{\alpha}\bar{\alpha}\psi_0\hat{\chi}\alpha+\bar{\alpha}\bar{\psi}_0\alpha\hat{\chi}\alpha+\bar{\alpha}\bar{\psi}_0\alpha\hat{\chi}\psi_0 \\ &\quad +\bar{\psi}_0(\bar{\alpha}\alpha)^2\hat{\chi}\psi_0+\bar{\alpha}(\bar{\psi}_0\alpha)^2\hat{\chi}\psi_0+\bar{\psi}_0(\bar{\alpha}\psi_0)^2\hat{\chi}\alpha \end{aligned} \tag{E.13}$$

$$\begin{aligned} B &= +\bar{\alpha}\alpha+(\bar{\alpha}\alpha)^2+\bar{\alpha}\psi_0\bar{\alpha}\alpha+\bar{\alpha}\psi_0+\bar{\psi}_0\psi_0(\bar{\alpha}\alpha)^2 \\ &\quad +\bar{\psi}_0\psi_0\bar{\alpha}\alpha+\bar{\psi}_0\alpha\bar{\alpha}\alpha+\bar{\alpha}\psi_0(\bar{\psi}_0\alpha)^2+\bar{\psi}_0\alpha(\bar{\alpha}\psi_0)^2+\bar{\psi}_0\alpha \end{aligned} \tag{E.14}$$

Without coefficients the matrices in the average are:

$$\begin{aligned} &= \bar{\psi}_0\hat{\chi}\psi_0+\bar{\alpha}\hat{\chi}\alpha+\bar{\psi}_0\hat{\chi}\alpha+\bar{\alpha}\bar{\alpha}\psi_0\hat{\chi}\alpha+\bar{\alpha}\bar{\psi}_0\alpha\hat{\chi}+\bar{\alpha}\bar{\psi}_0\alpha\hat{\chi}\psi_0\alpha \\ &\quad +\bar{\psi}_0(\bar{\alpha}\alpha)^2\hat{\chi}\psi_0+\bar{\psi}_0\hat{\chi}(\bar{\alpha}\psi_0)^2\alpha+\bar{\alpha}(\bar{\psi}_0\alpha)^2\hat{\chi}\psi_0+\bar{\alpha}\hat{\chi}\psi_0+\bar{\alpha}\alpha \\ &\quad +(\bar{\alpha}\alpha)^2+\bar{\alpha}\psi_0\bar{\alpha}\alpha+\bar{\alpha}\psi_0+\bar{\psi}_0\psi_0(\bar{\alpha}\alpha)^2+\bar{\psi}_0\psi_0\bar{\alpha}\alpha+\bar{\psi}_0\alpha\bar{\alpha}\alpha \\ &\quad +\bar{\alpha}\psi_0(\bar{\psi}_0\alpha)^2+\bar{\psi}_0\alpha(\bar{\alpha}\psi_0)^2+\bar{\psi}_0\alpha \end{aligned} \tag{E.15}$$

$$<\bar{\Psi}\hat{\chi}\Psi> \;=\; A+B+\bar{\psi}_0\hat{\chi}\psi_0 \tag{E.16}$$

$$\begin{aligned} <\bar{\Phi}\Phi> &= A+B-\bar{\psi}_0(\bar{\alpha}\alpha)^2\hat{\chi}\psi_0-\bar{\psi}_0(\bar{\alpha}\psi_0)^2\hat{\chi}\alpha+(\bar{\alpha}\psi_0)^2+(\bar{\psi}_0\alpha)^2 \\ &\quad +(\bar{\psi}_0\alpha)^2(\bar{\alpha}\psi_0)^2 \end{aligned} \tag{E.17}$$

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